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FOREWORD — AVANT-PROPOS — ПРЕДИСЛОВИЕ — PREFACIO

The Scientific Advisory Committee of the International Atomic Energy Agency in June 1959 recommended that the Agency undertake the promotion and exchange of information on plasma physics and controlled nuclear fusion research. As a specific proposal, the Committee recommended the establishment of an international scientific journal in this field. In response to this proposal the Agency now inaugurates "NUCLEAR FUSION."

We believe this new journal will contribute to the improvement of cooperation between nations in the difficult task of finding ways to extract energy from controlled nuclear fusion. Even if the earnest and concerted efforts devoted to this task do not lead immediately to practical energy sources, the endeavor will greatly expand the knowledge of plasma physics besides broadening basic scientific knowledge. The effort will also undoubtedly lead to the invention of new and important peacetime devices.

It is an unusual fact that, although the universe—as it is now understood—consists almost entirely of plasma, scientists have made progress only relatively recently in understanding the properties of this "fourth state of matter." As the pages of "NUCLEAR FUSION" show, the properties of plasma are not simple nor are they easily investigated.

All those who read and use "NUCLEAR FUSION" are invited to participate in the journal. Their comments and suggestions, as well as their manuscripts, will be welcomed. To those who have helped us in creating the journal, I wish to express, on behalf of the International Atomic Energy Agency, my sincere appreciation.

Le Comité consultatif scientifique de l'Agence internationale de l'énergie atomique recommanda en juin 1959 que l'Agence encourage la diffusion et l'échange de renseignements sur la recherche en matière de physique des plasmas et de fusion nucléaire contrôlée, notamment par la publication d'une revue scientifique internationale sur ces questions. Conformément à cette recommandations, l'Agence presente maintenant le premier numéro de sa revue «FUSION NUCLÉAIRE».

Nous sommes convaincus que cette nouvelle publication contribuera à resserrer la coopération entre les pays qui s'efforcent de résoudre le difficile problème de la production d'énergie à partir de la fusion nucléaire contrôlée. Même si leur effort intense et concerté n'aboutit pas immédiatement à la découverte de sources d'énergie utilisables, il n'en contribuera pas moins à étendre considérablement les connaissances en matière de physique des plasmas et à approfondir de plus les notions fondamentales de la science. Cet effort se traduira certainement aussi par l'invention de nouveaux et importantes dispositifs aux utilisations pacifiques.

Si l'univers — comme on le considère à present — se compose presque entièrement de plasma, il est assez étonnant que les hommes de science n'aient fait que des progres relativement recents dans la comprehension des propriétés de ce qui peut être considéré comme le «quatrième état de la matière». Comme il apparaîtra à la lecture de «FUSION NUCLÉAIRE», ces propriétés ne sont ni simples ni d'une étude facile.

Tous les lecteurs de «FUSION NUCLÉAIRE» sont invités y collaborer, à adresser à l'éditeur leurs observaions, leurs suggestions ainsi que les communications qu'ils voudraient y voir inséré. A tous ceux qui nous ont aidés à mettre au point cette revue, je tiens à adresser, au nom de l'Agence internationale de l'énergie atomique, mes emerciements les plus sincères.

В июне 1959 года Научно-консультативный комитет Международного агентства по атомной энергии рекомендовал Агентству принять меры, способствующие обмену информацией по исследованиям в области физики плазмы и контролируемого синтеза. В качестве конкретного предложения Комитет рекомендовал выпуск международного научного журнала в этой области. В соответствии с этим предложением Агентство в настоящее время приступает к изданию журнала «Ядерный синтез».

Мы считаем, что этот новый журнал будет содействовать улучшению сотрудничества между странами в решении трудной задачи нахождения путей получения энергии из контролируемого ядерного синтеза. Даже если серьезные и согласованные усилия, направленные на решение этой задачи, сразу и не подведут к практическим источникам энергии, эта попытка, помимо углубления основных научных знаний, значительно расширит познания в области физики плазмы. Это усилие несомненно приведет также к новым и важным техническим изобретениям мирного характера.

Необыкновенный факт заключается в том, что, хотя космос — как он понимается в настоящее время — почти полностью состоит из плазмы, ученые лишь сравнительно недавно достигли прогресса в понимании свойств этого «четвертого состояния материи». Как показывают опубликованные в журнале «Ядерный синтез» статьи, плазма обладает отнюдь не простыми свойствами, и нелегко поддаются эти свойства исследованию.

Все читатели журнала «Ядерный синтез» приглашаются участвовать в его издании. Их замечания, предложения, а также статьи будут приветствоваться. От имени Международного агенства по атомной энергии я хотел бы выразить мою искреннюю благодарность всем тем, кто помог нам в создании этого журнала.

En junio de 1959, el Comité Consultivo Científico del Organismo Internacional de Energía Atómica recomendó que el OIEA fomentase la difusión y el intercambio de información en el terreno de las investigaciones sobre la fusión nuclear controlada y la física del plasma. Además, el Comité recomendó concretamente que se iniciase la publicación de una revista científica internacional que tratase de estas cuestiones. Como consecuencia de esta propuesta, el Organismo lanza hoy el primer número de "FUSIÓN NUCLEAR".

Creemos que esta nueva revista contribuirá a acentuar la cooperación entre las diversas naciones en la difícil labor de descubrir procedimientos para la obtención de energía partiendo de la fusión nuclear controlada. Aun cuando los más serios y armonizados esfuerzos que se dediquen a esta labor no conduzcan inmediatamente a disponer de fuentes energéticas, la empresa extenderá muy considerablemente el conocimiento de la física del plasma además de ampliar el caudal de conocimientos científicos básicos. Estos esfuerzos se traducirán sin duda alguna en la invención de nuevos e importantes dispositivos utilizables en tiempo de paz.

Es realmente sorprendente el hecho de que, aunque el universo — según el concepto que hoy se tiene del mismo — está formado casi enteramente por plasma, hasta hace relativamente poco no hayan logrado los hombres de ciencia progresar en el conocimiento de las propiedades de este "cuarto estado de la materia". Como se refleja en las páginas de "FUSIÓN NUCLEAR", la investigación de las propiedades del plasma no es labor fácil ni sencilla.

Quienquiera que lea y haga uso de "FUSIÓN NUCLEAR" queda invitado a colaborar en la revista. Sus observaciones y sugerencias, así como los originales que envíe, serán bien acogidos. En cuanto a quienes nos ayudaron a lanzar esta revista, deseo expresarles aquí, en nombre del Organismo Internacional de Energía Atómica, mi sincero agradecimiento.

Sterling Cole

Director General

NUCLEAR FUSION: 1, 3-41 (1960)

PLASMA OSCILLATIONS (I)*

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This article is the first part of a review of the theory of wave phenomena in plasmas. The basic kinetic theory is developed taking into account coulomb collisions. From this magnetohydrodynamic equations are derived. It is demonstrated that it is legitimate to deal with a closed subset of these equations either in the limit of collision-dominated phenomena, or in the limit where the effective phase velocities of the phenomena of interest are much greater than thermal speeds. In case of collision-dominated plasmas the theory of transport coefficients is discussed.

These equations are then applied to an extensive treatment of small amplitude wave phenomena in plasmas. A discussion of the dissipative effects on hydromagnetic waves is given. Hydromagnetic waves are also considered from the Chew, Goldberger and Low theory. Longitudinal and transverse oscillations in current-carrying plasmas are also discussed.

Oscillations of a cylindrical plasma are considered and the phenomenon of ion cyclotron resonance is discussed. The possibility of radiation by plasma oscillations by a uniform sphere is exhibited. Some general results on the stability of longitudinal electron oscillations in non-uniform plasmas are given.

A brief treatment of large amplitude electron oscillations is given and the breaking of these oscillations as a dissipative mechanism for the organized plasma motion is discussed.

Part II of this paper, to appear in a future issue of this journal, will be devoted to the discussion of plasma oscillations directly from the kinetic theory.

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I. Introduction

In this article we shall attempt to present a systematic survey of the theoretical knowledge of wave phenomena in plasmas. We shall use the term "plasma" to describe a fully ionized gas composed of positive and negative ions of effectively zero net charge density. For the sake of convenience we limit our attention to a plasma composed of only two species (positive ions and electrons); the extension of the results to gases consisting of more than two species is straightforward and will not be considered here. Some of the results are also applicable to conducting liquids and gases not fully ionized.

Astrophysical examples of such systems are the ionosphere, the solar corona, stellar interiors and the atmosphere of hot stars, the H_{II} regions of interstellar space, and cosmic radio sources. Plasmas are also of fundamental interest in the proposed schemes for controlled thermonuclear fusion, plasma propulsion of space ships, microwave generators, and gas discharge devices.

One distinctive feature of plasmas is the special nature of collisions, in contrast to the two-body collisions familiar in the conventional theory of neutral gases. Because of the long range nature of the Coulomb forces which govern the encounters of charged particles, cooperative phenomena are of major importance in describing the dynamics of a plasma. Also these long range collisions lead to a diffusion in velocity space and are most appropriately described by the Fokker-Planck equation. The elastic or inelastic two-body encounters of ions or electrons with neutrals or perhaps with the core of an incompletely ionized atom are usually negligible in plasmas and will be considered, effectively, only when dealing with dissipative effects in collision-dominated plasmas.

There are two levels of description of plasmas which have received major consideration during the past decade. The first of these is based on the direct solution of the Boltzmann transport equation—usually referred to as the kinetic theory description. This method has the intrinsic merit of providing a more complete description of the dynamics of a plasma. However, this procedure is mathematically more difficult to handle. The second method is based on using a closed set of moment equations to characterize the behaviour of a plasma. This description, usually referred to as the hydrodynamic or hydromagnetic description, though of limited applicability (in a sense to be discussed in detail later) has the merit of relative mathematical simplicity and is quite helpful in the physical understanding of the various phenomena involved. This method is applicable either when (1) collisions dominate so that one has local thermodynamic equilibrium and can apply the Chapman-Enskog expansion technique, or (2) the phase velocity of the wave is much larger than the characteristic thermal speed in a collision-free plasma. The latter will be referred to as the "Low Temperature Approximation."

For plasmas of interest under astrophysical and laboratory conditions quantum mechanical effects are unimportant in considering most of the dynamics of plasmas.

This work, therefore, divides itself naturally into two parts. In part I the kinetic theory description is formulated and its relation to the hydrodynamic description established. The resulting hydromagnetic equations are then applied to a variety of wave phenomena. Part II of this article, to appear in a later issue of this journal, will be devoted to the discussion of wave phenomena based directly on the kinetic theory.

The theory described here is concerned mainly with the small motions about a steady state of the plasma. The general case of large motions is considered only briefly in Section V.

Section III deals with oscillations in unbounded homogeneous plasmas, admittedly a severe idealization. However, these treatments do serve to give some insight into the basic physical processes and the mathematical methods. The neglect of boundaries would imply that the description is valid only for wavelengths much smaller than the characteristic length of the system and for time intervals short enough that the disturbance has not reached the boundaries and been reflected and transmitted.

Unfortunately, because of the inherent difficulty of establishing in the laboratory plasmas uncomplicated by boundary effects, impurities, etc., there has not been, as yet, much experimental verification of the theoretical investigations.

II. The basic equations

By and large one is interested only in the knowledge of some macroscopic properties of the plasma like the mean particle density, mean velocity, etc., rather than in the detail afforded by the description of the individual motion of the particles. It is, therefore, appropriate to characterize a given class of particles by a distribution function $f(\mathbf{x},\mathbf{v},t)$ such that $f\,d^3\mathbf{x}\,d^3\mathbf{v}$ represents at time t the probable number of particles with velocities between $\mathbf{v}$ and $\mathbf{v}+d\mathbf{v}$, and with positions between $\mathbf{x}$ and $\mathbf{x}+d\mathbf{x}$. It is possible to relate f to the more recondite density in the phase space of all the particles, but a discussion of this matter is beyond the scope of this article; suffice it to say that the introduction of f is compatible with the following notions about particle interactions in plasmas.

The particles, being charged and possessing no internal degrees of freedom at the range of energies of interest here, interact primarily through the electromagnetic fields. In particular, when two particles are close together it is their Coulomb interaction which predominates. Consider then such two-body Coulomb encounters. They can be characterized, in their center of mass system, by the impact parameter and the relative velocity. The collisions can then be conveniently divided into three classes.

The first of these corresponds to impact parameters less than the mean impact parameter for 90° deflection,

$$b = \frac{Z_1 Z_2 e^2}{\langle v^2 \rangle}\left[\frac{1}{m_1}+\frac{1}{m_2}\right], \qquad (1)$$

where Z_1 and Z_2 are the charge numbers of particles involved, m_1 and m_2 are their masses, $\langle v^2 \rangle$ represents their mean square relative velocity, and e is the electronic charge. It is usually legitimate to describe such encounters by the familiar two-body collision integrals of the kinetic theory of gases when writing the kinetic equation for the distribution function. In most cases of interest these collisions are negligible.

The second class comprises the many-body encounters which can be regarded as composed of a succession of uncorrelated small angle Coulomb deflections. These correspond to elementary Coulomb scattering events characterized by impact parameters greater than the distance of closest approach b, but less than the electron Debye length

$$\lambda_D = \left[\frac{(1/3)\, m \langle v^2 \rangle}{4\pi N e^2}\right]^{\frac{1}{2}} \qquad (2)$$

where m is the electron mass, $\langle v^2 \rangle$ is the electron mean square velocity, and N is the electron number density. These encounters are most appropriately described by the Fokker-Planck form for the collision integral in the Boltzmann equation.

When there are many particles in a sphere of radius λ_D (i.e., $\lambda_D N \gg 1$), density fluctuations over distances $> \lambda_D$ are suppressed by macroscopic electric forces. Thus the third class of uncorrelated two-body encounters, those with impact parameters greater than λ_D, do not occur per se. These are replaced in the kinetic eqation by the correlated effect of many particles through the macroscopic electromagnetic fields described by the Maxwell equations; this leads to co-operative phenomena.

1. Boltzmann Equation

The distribution function for each species of particles is governed by the Boltzmann equation

$$\mathfrak{D} f \equiv \left[\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla + \mathbf{F} \cdot \nabla_v\right] f = \left(\frac{\partial f}{\partial t}\right)_{\text{coll.}} \qquad (3)$$

where ∇ and ∇_v represent the gradient operators in co-ordinate ($\mathbf{x}$) and velocity ($\mathbf{v}$) space respectively, $\mathbf{F}$ denotes the force per unit mass, and $(\partial f / \partial t)_{\text{coll.}}$ represents the time rate of change of f resulting from collisions. It may be remarked here that there is a separate Boltzmann equation for each species of particle. We now discuss the Fokker-Planck form for the collision term.

1.1 Fokker-Planck Form of the Boltzmann Equation

It has been demonstrated by Jeans that when particles interact through long range forces, the cumulative effect of weak deflections resulting from the relatively distant encounters is more important than the effect of occasional large deflections. The cumulative effect of weak deflections is most appropriately taken into account by writing the collision term in the Fokker-Planck form. This reduction is possible only if there exist time intervals Δt long enough for the particles to suffer a large number of collisions but small enough for the net mean square change in velocity, $\langle |\Delta \mathbf{v}|^2 \rangle$, to be small compared with the mean square velocity, and to be of order Δt.

Let $P(\mathbf{v}; \Delta\mathbf{v})$ denote the probability that in the time interval Δt a particle with velocity co-ordinate $\mathbf{v}$ undergoes a displacement $\Delta\mathbf{v}$. The distribution function is then given by

$$f(\mathbf{x}, \mathbf{v}, t) =$$
$$\int f(\mathbf{x}, \mathbf{v} - \Delta\mathbf{v}, t - \Delta t)\, P(\mathbf{v} - \Delta\mathbf{v}; \Delta\mathbf{v})\, d^3 \Delta\mathbf{v}. \qquad (4)$$

We may remark here that in expecting this integral equation to be true we are actually supposing that the course which a particle will take depends only on the instantaneous value of its physical parameters and is entirely independent of its past history. In general probability theory a stochastic process which has this characteristic, namely that what happens at a given instant of time t depends only on the state of the system at time t, is said to be a Markoff process.

Since Δt and $\Delta\mathbf{v}$ are both assumed small, using Taylor's expansion we can write (4) in the form

$$f(\mathbf{x}, \mathbf{v}, t) =$$
$$\int \Big\{ f(\mathbf{x}, \mathbf{v}, t)\, P(\mathbf{v}; \Delta\mathbf{v}) - \Delta \mathrm{t} \frac{\partial f}{\partial t} P(\mathbf{v}; \Delta\mathbf{v})$$
$$- \Delta\mathbf{v} \cdot \left[\frac{\partial f}{\partial \mathbf{v}} P(\mathbf{v}; \Delta\mathbf{v}) + f \frac{\partial}{\partial \mathbf{v}} P(\mathbf{v}; \Delta\mathbf{v})\right] + \frac{1}{2} \Delta\mathbf{v}\, \Delta\mathbf{v} : \qquad (5)$$
$$\left[\frac{\partial^2 f}{\partial \mathbf{v}\, \partial \mathbf{v}} P(\mathbf{v}; \Delta\mathbf{v}) + 2 \frac{\partial f}{\partial \mathbf{v}} \frac{\partial P}{\partial \mathbf{v}} + f \frac{\partial^2 P}{\partial \mathbf{v}\, \partial \mathbf{v}}\right] + \ldots \Big\}\, d^3 \Delta\mathbf{v}.$$

Using the fact that

$$\int P(\mathbf{v}; \Delta\mathbf{v})\, d^3 \Delta\mathbf{v} = 1, \qquad (6)$$

Eq. (5) gives for the time rate of change of f resulting from the cumulative effect of small deflections:

$$\Delta t \left(\frac{\partial f}{\partial t}\right)_{\text{coll}} = - \frac{\partial}{\partial \mathbf{v}} \cdot \left[\langle \Delta\mathbf{v} \rangle - \frac{1}{2} \frac{\partial}{\partial \mathbf{v}} \cdot \langle \Delta\mathbf{v}\, \Delta\mathbf{v} \rangle\right], \qquad (7)$$

where we have retained terms only to lowest significant order and

$$\langle \Delta\mathbf{v} \rangle = \int f(\mathbf{x}, \mathbf{v}, t)\, P(\mathbf{v}; \Delta\mathbf{v})\, \Delta\mathbf{v}\, d^3 \Delta\mathbf{v}, \qquad (8)$$

$$\langle \Delta\mathbf{v}\, \Delta\mathbf{v} \rangle = \int f(\mathbf{x}, \mathbf{v}, t)\, P(\mathbf{v}; \Delta\mathbf{v})\, \Delta\mathbf{v}\, \Delta\mathbf{v}\, d^3 \Delta\mathbf{v}. \qquad (9)$$

Eq. (7) represents the collision term in the Fokker-Planck form.

Consider the two-body encounters of particles of type i with those of type j and let $\sigma_{ij}(\theta, \mathbf{v})$ denote the differential cross section for scattering of their relative velocity vector $\mathbf{v} = \mathbf{v}_i - \mathbf{v}_j$ through an angle θ. Then the associated momentum transfer cross section is given by

$$Q_{ij}(v) = 2\pi \int \sigma_{ij}(\theta, v)\, (1 - \cos\theta) \sin\theta\, d\theta. \qquad (10)$$

In terms of Q, retaining only the dominant terms in the mean square scattering angle, one can write for the contribution of such two-body encounters to stochastic averages, when polarization effects are negligible,

$$\langle \Delta\mathbf{v} \rangle_{ij} \qquad (11)$$
$$= f_i(\mathbf{x}, \mathbf{v}_i, t) \frac{m_i}{(m_i + m_j)} \int d^3 \mathbf{v}_j f_j(\mathbf{x}, \mathbf{v}_j, t)\, v\, Q_{ij}(v)\, \nabla_v v,$$

and

$$\langle \Delta\mathbf{v}\, \Delta\mathbf{v} \rangle_{ij} = f(\mathbf{x}, \mathbf{v}_i, t) \frac{m_i^2}{(m_i + m_j)^2}$$
$$\int d^3 \mathbf{v}_j f_j(\mathbf{x}, \mathbf{v}_j, t)\, v^3 Q_{ij}(v)\, \nabla_v \nabla_v v. \qquad (12)$$

1.2 Fokker-Planck Equation for Coulomb Force

For the case of Coulomb collisions, the differential scattering cross section is given by

$$\sigma_{ij}(v) = \left[\frac{Z_i Z_j e^2 (m_i + m_j)}{2\, m_i\, m_j\, v^2}\right]^2 \frac{1}{\sin^4(\theta/2)}. \qquad (13)$$

The integral occurring in Q will diverge at the lower limit of integration ($\theta = 0$). However, the small angle deflections correspond to scatterings with very large impact parameters, and the divergence arises from the long range nature of the Coulomb force. This divergence is eliminated when we take into account the shielding that arises from the polarization of charge surrounding the scatterer. The polarization screens the scattering particle and provides a natural cutoff on the maximum impact parameter of the order of the Debye length λ_D. We thus take for the minimum scattering angle

$$\psi = \frac{(m_i + m_j)\, e^2}{4\, m_i\, m_j \langle v^2 \rangle \lambda_D}. \qquad (14)$$

Using Eqs. (13) and (14) into Eq. (10) we obtain

$$Q_{ij} = 4\,\pi \left[\frac{Z_i Z_j e^2 (m_i + m_j)}{m_i\, m_j\, v^2}\right]^2 \ln \frac{(m_i\, m_j)\, e^2}{2\, m_i\, m_j \langle v^2 \rangle \lambda_D}. \qquad (15)$$

Using the foregoing results in the Fokker-Planck equation we obtain after an integration by parts what is called the Landau form for the collision term:

$$\left(\frac{\partial f}{\partial t}\right)_{\text{coll.}} = \nabla_{v_i} \cdot \left\{\frac{2\pi\, Z_i^2\, Z_j^2\, e^4}{m_i^2\, m_j^2} \ln\left[\frac{(m_i + m_j)\, e^2}{2 m_i\, m_j \langle v^2 \rangle_{ij} \lambda_D}\right]\right.$$
$$\left. \times \int d^3 v_j \frac{v^2 \mathbf{I} - \mathbf{v}\mathbf{v}}{v^3} \cdot (m_j f_j \nabla_{v_i} f_i - m_j f_i \nabla_{v_j} f_j)\right\}. \qquad (16)$$

It may be noted that, apart from the slowly varying logarithmic term, in order of magnitude the collision terms are inversely proportional to $\langle v^2 \rangle_{ij}$, the mean square relative kinetic energy. Thus in the limit where this factor is appropriately large, collisions are negligible.

1.3 The Moment Equations

The particle density and the local mean velocity are defined as

$$N = \int f(\mathbf{x}, \mathbf{v}, t)\, d^3\mathbf{v} \qquad (17)$$

and

$$\mathbf{u} = \frac{1}{N} \int f(\mathbf{x}, \mathbf{v}, t)\, \mathbf{v}\, d^3\mathbf{v}. \qquad (18)$$

In terms of these moments we can define some of the other relevant macroscopic quantities*:

$$\varrho = \sum m\, N = \text{mass density}, \qquad (19)$$

$$\mathbf{U}_0 = \frac{1}{\varrho} \sum m\, N\, \mathbf{u} = \text{velocity of the center of mass}, \qquad (20)$$

$$\varepsilon = \sum e\, N = \text{charge density}, \qquad (21)$$

$$\mathbf{J} = \sum e\, N\, \mathbf{u} = \text{current density}, \qquad (22)$$

where the summation is to be carried over all the species of the system. The random velocity with respect to the center of mass is defined to be

$$\mathbf{w} = \mathbf{v} - \mathbf{U}_0. \qquad (23)$$

It is convenient to define the other relevant quantities with respect to the center of mass velocity. Thus

$$\mathbf{U} = \frac{1}{N} \int \mathbf{w}\, f(\mathbf{x}, \mathbf{v}, t)\, d^3\mathbf{v} \qquad \begin{array}{l}= \text{mean velocity of} \\ \text{each of the species} \\ \text{with respect to the} \\ \text{center of mass,}\end{array} \qquad (24)$$

$$\mathsf{P} = m \int \mathbf{w}\,\mathbf{w}\, f(\mathbf{x}, \mathbf{v}, t)\, d^3\mathbf{v} \qquad = \text{pressure tensor}, \qquad (25)$$

and

$$\mathsf{Q} = m \int \mathbf{w}\,\mathbf{w}\,\mathbf{w}\, f(\mathbf{x}, \mathbf{v}, t)\, d^3\mathbf{v} = \text{heat flow tensor}, \qquad (26)$$

where Q is a tensor of rank 3 completely symmetric in all its indices. We further define

$$\boldsymbol{\mathsf{P}} = \sum \mathsf{P} \text{ and } \boldsymbol{\mathsf{Q}} = \sum \mathsf{Q}. \qquad (27)$$

It is clear that

$$\sum m\, N\, \mathbf{U} = 0. \qquad (28)$$

Note that

$$\frac{1}{2} \text{Trace } \mathsf{P} = \int \frac{1}{2} m\, w^2 f(\mathbf{x}, \mathbf{v}, t)\, d^3\mathbf{v} \qquad (29)$$

represents the mean kinetic energy relative to the center of mass and is referred to as the internal energy, or the thermal energy of the gas. This suggests that we can define the generalized temperature by the relation

$$\frac{1}{2} \text{Trace } \mathsf{P} = \frac{3}{2} N \Theta, \qquad (30)$$

where $\Theta = kT$, k is the Boltzmann constant, and T is the temperature. We shall always use Θ to denote the temperature in energy units.

This definition of temperature is natural when collisions dominate, in which case the various particle distribution functions are very closely locally Maxwellian, relative to the center of mass velocity, corresponding to the same temperature. Occasionally one encounters an alternative definition of temperature, where it is defined separately for each species of the plasma, as the mean kinetic energy of that species relative to its own mean mass velocity. The other moments can be similarly defined. This latter definition is particularly convenient when collisions are negligible and will be used in part II of this review. The two notions, of course, coincide if we consider the motion of a single species of the plasma, a picture which is reasonable in investigating high frequency waves where only the electron motion is significant. In general, however, the two definitions are quite different and care must be exercised to make sure which is being employed in any given context.

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* We shall adopt the convention that whenever m appears after a summation sign, it is the mass of any of the species of the plasma; otherwise m denotes the electron mass and M, the proton mass. The electron charge is taken to be $-e$ while that of the proton, $+e$. Similarly, ω_c will denote the electron or ion cyclotron frequency whereas ω_e and ω_i will denote the electron and ion cyclotron frequency respectively.

The vector

$$\mathbf{q}=\frac{1}{2}\,\mathbf{I}:\mathsf{Q} \qquad \int \frac{1}{2}\,m\,w^2\,\mathbf{w}\,f\,(\mathbf{x},\mathbf{v},t)\,d^3\,\mathbf{v} \qquad (31)$$

represents the flow of internal energy relative to the center of mass and will be called the heat flow vector.

Since $\mathbf{x}$ and $\mathbf{v}$ are independent vectors, it is convenient to write the Boltzmann equation (3) as

$$\mathfrak{D}f=\frac{\partial f}{\partial t}+\nabla\cdot(\mathbf{v}f)+\nabla_v\cdot(\mathbf{F}\,f)=\left(\frac{\partial f}{\partial t}\right)_{\text{coll}} \qquad (32)$$

Quite generally, we can write

$$\mathbf{F}=\frac{e}{m}\left[\mathbf{E}\,(\mathbf{x},t)+\frac{1}{c}\,\mathbf{v}\times\mathbf{B}\,(\mathbf{x},t)\right]-\nabla\,\varphi, \qquad (33)$$

where $\mathbf{E}$ and $\mathbf{B}$ denote the electric and magnetic fields respectively, and φ is the gravitational potential. The electromagnetic fields, of course, satisfy Maxwell's equations:

$$\nabla\times\mathbf{B}=\frac{4\,\pi}{c}\,\mathbf{J}+\frac{1}{c}\,\frac{\partial\mathbf{E}}{\partial t}, \qquad \nabla\times\mathbf{E}=-\frac{1}{c}\,\frac{\partial\mathbf{B}}{\partial t}, \qquad (34)$$

$$\nabla\cdot\mathbf{E}=4\,\pi\,\varepsilon, \qquad \text{and} \qquad \nabla\cdot\mathbf{B}=0; \qquad (35)$$

Gaussian units will be used throughout in this work. The gravitational potential satisfies the Poisson equation

$$\nabla^2\varphi=-4\,\pi\,G\,\varrho, \qquad (36)$$

where G denotes the gravitational constant.

In order to derive the equations governing the macroscopic quantities, it is convenient to write the Boltzmann equation in terms of the random velocity $\mathbf{w}$. This can be readily effected by the transformations

$$f\,(\mathbf{x},\mathbf{v},t)\rightarrow f\,(\mathbf{x},\mathbf{w},t), \qquad (37)$$

$$\frac{\partial}{\partial t}\rightarrow\frac{\partial}{\partial t}-\frac{\partial\,\mathbf{U}_0}{\partial t}\cdot\nabla_w\,, \qquad (38)$$

$$\nabla\rightarrow\nabla-(\nabla\,\mathbf{U}_0)\cdot\nabla_w\,, \qquad (39)$$

$$\nabla_v\rightarrow\nabla_w\,. \qquad (40)$$

The Boltzmann equation now takes the form

$$\mathfrak{D}f=\frac{\partial f}{\partial t}+\nabla\cdot\Big[\,(\mathbf{w}+\mathbf{U}_0)f\,\Big]+\nabla_w\cdot(\mathbf{b}f) \qquad \left(\frac{\partial f}{\partial t}\right)_{\text{coll}} \quad (41)$$

where

$$=\frac{e}{m}\left[\mathbf{E}\,(\mathbf{x},t)+\frac{1}{c}\,\mathbf{U}_0\times\mathbf{B}\,(\mathbf{x},t)\right]$$
$$-\nabla\,\varphi-\frac{d\,\mathbf{U}_0}{dt}-(\mathbf{w}\cdot\nabla)\,\mathbf{U}_0+\frac{e}{mc}\,\mathbf{w}\times\mathbf{B}\,(\mathbf{x},t), \quad (42)$$

nd

$$\frac{d}{dt}=\frac{\partial}{\partial t}+\mathbf{U}_0\cdot\nabla \qquad (43)$$

enotes the time derivative as we follow the motion of n element of the fluid.

Multiplying Eq. (41) by 1, $m\mathbf{w}$, and $m\mathbf{w}\mathbf{w}$, and asuming that f vanishes sufficiently strongly in velocity pace so that all the surface integrals vanish, we obtain r the moment equations

$$\frac{\partial N}{\partial t}+\nabla\cdot\Big[N\mathbf{U}+N\mathbf{U}_0\Big]=0\,, \qquad (44)$$

$$mN\frac{d\,\mathbf{U}_0}{dt}=-\nabla\cdot\Big[\mathsf{P}+mN\mathbf{U}_0\mathbf{U}\Big]$$
$$-\frac{\partial}{\partial t}\,m\,N\,\mathbf{U}+N\,e\left[\mathbf{E}+\frac{1}{c}\,\mathbf{U}_0\times\mathbf{B}\right]$$
$$-\,m\,N\,\nabla\,\varphi\,-\,m\,N\,(\mathbf{U}\cdot\nabla)\,\mathbf{U}_0+\frac{Ne}{c}\,\mathbf{U}\times\mathbf{B}\,, \qquad (45)$$

and

$$\frac{\partial}{\partial t}\mathsf{P}+\nabla\cdot(\mathsf{Q}+\mathbf{U}_0\,\mathsf{P})+\mathsf{P}\cdot\nabla\,\mathbf{U}_0+(\mathsf{P}\cdot\nabla\,\mathbf{U}_0)^T$$
$$+\frac{e}{m\,c}\,(\mathbf{B}\times\mathsf{P}-\mathsf{P}\times\mathbf{B})$$
$$-\,m\,N\left[\frac{e}{m}\,(\mathbf{E}+\frac{1}{c}\,\mathbf{U}_0\times\mathbf{B})-\nabla\,\varphi\right]\mathbf{U}$$
$$-\,mN\,\mathbf{U}\left[\frac{e}{m}\,(\mathbf{E}+\frac{1}{c}\,\mathbf{U}_0\times\mathbf{B})-\nabla\,\varphi\right]$$
$$+\,m\,N\left[\frac{d\,\mathbf{U}_0}{d\,t}\,\mathbf{U}+\mathbf{U}\,\frac{d\,\mathbf{U}_0}{d\,t}\right]=\left(\frac{\partial}{\partial\,t}\,\mathsf{P}\right)_{\text{coll.}} \qquad (46)$$

where $(\mathsf{A})^T$ denotes the transpose of the matrix A. We now sum Eqs. (44)—(46) over all the species of the system and, making use of Eqs. (19)—(22), (27), and (28), we obtain

$$\frac{\partial\,\varrho}{\partial\,t}+\nabla\cdot(\varrho\,\mathbf{U})=0\,, \qquad (47)$$

$$\varrho\,\frac{d\mathbf{U}}{dt} \qquad -\nabla\cdot\mathbf{P}+\varepsilon\,\mathbf{E}+\frac{1}{c}\,\mathbf{J}\times\mathbf{B}-\varrho\,\nabla\varphi, \quad (48)$$

and

$$\frac{\partial}{\partial t}\mathbf{P}+\nabla\cdot(\mathbf{Q}+\mathbf{U}\,\mathbf{P})+\mathbf{P}\cdot\nabla\mathbf{U}+(\mathbf{P}\cdot\nabla\mathbf{U})^T$$
$$+\frac{e}{m\,c}\,(\mathbf{B}\times\mathbf{P}-\mathbf{P}\times\mathbf{B})-(\mathbf{J}-\varepsilon\,\mathbf{U})\,(\mathbf{E}+\frac{1}{c}\,\mathbf{U}\times\mathbf{B})$$
$$-\,(\mathbf{E}+\frac{1}{c}\,\mathbf{U}\times\mathbf{B})\,(\mathbf{J}-\varepsilon\,\mathbf{U})=\left(\frac{\partial}{\partial\,t}\,\mathbf{P}\right)_{\text{coll.}} \qquad (49)$$

where in writing Eq. (49) we have made use of the fact that

$$\sum e\,N\,\mathbf{U}=\sum e\,N\,(\mathbf{U}-\mathbf{U}_0)=\mathbf{J}-\varepsilon\,\mathbf{U}_0, \qquad (50)$$

and we have dropped the subscript on $\mathbf{U}_0$ in Eqs. (47)—(49) as it is no longer necessary. If we take the trace of Eq. (49) and make use of the relation (31), we obtain

$$\frac{\partial}{\partial\,t}\left(\frac{3}{2}\,n\,\Theta\right)+\nabla\cdot\left(\mathbf{q}+\frac{3}{2}\,n\,\Theta\,\mathbf{U}\right)+\mathbf{P}:\nabla\mathbf{U}$$
$$-\,(\mathbf{J}-\varepsilon\,\mathbf{U})\cdot\left(\mathbf{E}+\frac{1}{c}\,\mathbf{U}\times\mathbf{B}\right)=0, \qquad (51)$$

where the right hand side vanishes since collisions conserve the internal energy of the system.

We recognize Eqs. (47) and (48) as the well-known hydrodynamical equations of continuity and motion. Eqs. (47)—(49) are rather useless in their present form; they do not constitute a closed set of equations as they contain more unknowns than the number of equations. One could obtain an equation governing the heat flow tensor $\mathbf{Q}$ by taking the fourth moment of the Boltzmann equation; however, this equation would involve the next higher order moment of the distribution function and the system of equations is not closed again.

THE CASE OF STRONG COLLISIONS

We now consider the case where collisions dominate. Then, to a high degree of approximation, the particle distribution functions tend to be locally Maxwellian and we may set

$$f(\mathbf{x}, \mathbf{v}, t) = f_0(\mathbf{x}, \mathbf{v}, t)\,[1 + \Phi(\mathbf{x}, \mathbf{v}, t)], \tag{52}$$

where

$$f_0(\mathbf{x}, \mathbf{v}, t) = N(\mathbf{x}, t)\left[\frac{m}{2\pi\Theta(\mathbf{x}, t)}\right]^{3/2} \exp\left\{-m\,\frac{[\mathbf{v} - \mathbf{U}_0(\mathbf{x}, t)]^2}{2\Theta(\mathbf{x}, t)}\right\}, \tag{53}$$

and Φ is considered small compared to unity. The density N, temperature Θ, and mass velocity $\mathbf{U}_0$ are assumed to be slowly-varying functions of space and time such that their spatial variations over one mean free path, and temporal variations over one mean free time, are very small. This limiting situation forms the basis of the Chapman-Enskog theory of gases and leads to a truncation of the hierarchy of moment equations.

The rigorous Chapman-Enskog theory for a plasma composed of two or more species, interacting with electromagnetic and gravitational fields, is quite complex. However, the smallness of the electron to ion mass ratio and the inability of a plasma to support large electric fields—referred to as the condition of quasi-neutrality—results in a substantial simplification of the Chapman-Enskog expansion technique.

The condition of quasi-neutrality implies that the ratio of the net charge density to the charge density of each of the species is very small compared to unity. We shall systematically neglect both these ratios compared with unity in our discussion of the transport co-efficients.

Consider for simplicity a plasma composed of N electrons and an equal number of protons of masses m and M and charges $-e$ and $+e$ respectively. Following the Chapman-Enskog procedure we can show that, within the approximation outlined earlier, the small quantities Φ are determined by the integro-differential equations (cf. Eqs. (3), (16), (52), (53)):

$$\begin{aligned}&\left\{\left(\frac{Mw^2}{2\Theta} - \frac{5}{2}\right)\mathbf{w}\cdot\nabla\ln\Theta + \left(\frac{M}{\Theta}\,\mathbf{w}\,\mathbf{w} - \frac{1}{3}\,\mathbf{w}^2\,\mathbf{I}\right) : \nabla\mathbf{U}_0 \right.\\ &\left. - \left[\frac{e}{\Theta}\left(\mathbf{E} + \frac{1}{c}\,\mathbf{U}_0\times\mathbf{B}\right) + \nabla\ln(N\Theta)\right]\cdot\mathbf{w}\right\} f_0^+ = \\ &-\frac{1}{N\Theta c}(\mathbf{J}\times\mathbf{B})\cdot\mathbf{w} f_0^+ + \frac{e}{Mc}\mathbf{B}\cdot\mathbf{w}\times(\nabla_w\Phi^+) f_0^+ - \nabla_{w'}\Gamma^+\end{aligned} \tag{54}$$

and

$$\begin{aligned}&\left\{\left(\frac{mw^2}{2\Theta} - \frac{5}{2}\right)\mathbf{w}\cdot\nabla\ln\Theta + \frac{m}{\Theta}\left(\mathbf{w}\,\mathbf{w} - \frac{1}{3}\,\mathbf{w}^2\,\mathbf{I}\right) : \nabla\mathbf{U}_0 \right.\\ &\left. + \left[\frac{e}{\Theta}\left(\mathbf{E} + \frac{1}{c}\,\mathbf{U}_0\times\mathbf{B}\right) + \nabla\ln(N\Theta)\right]\cdot\mathbf{w}\right\} f_0^- = \\ &-\frac{e}{mc}\mathbf{B}\cdot\mathbf{w}\times(\nabla_w\Phi^-) f_0^-\cdot\nabla_w\cdot\Gamma^- .\end{aligned} \tag{55}$$

The currents in velocity space Γ, obtained by an appropriate expansion of the Fokker-Planck Eq. (16), are given by

$$\begin{aligned}-\Gamma^+ = \frac{2\pi e^4\ln\Lambda}{M^2}&\left\{\int\!\!\int d^3\mathbf{w}' f_0^+(\mathbf{w}) f_0^+(\mathbf{w}')\,\frac{\mathbf{g}^2\mathbf{I} - \mathbf{g}\,\mathbf{g}}{g^3}\,\cdot\right.\\ &\left[\nabla_w\Phi^+(\mathbf{w}) - \nabla_{w'}\Phi^+\right] - \frac{M}{m}\int d^3\mathbf{w} f_0^+(\mathbf{w}) f_0^-(\mathbf{w}')\\ &\left.\frac{\mathbf{w}^2\mathbf{I} - \mathbf{w}'\,\mathbf{w}'}{\mathbf{w}'^3}\cdot\nabla_w\Phi^-(\mathbf{w}')\right\}\end{aligned} \tag{56}$$

and

$$\begin{aligned}-\Gamma^- = \frac{2\pi e^4\ln\Lambda}{m^2}&\left\{\int\!\!\int d^3\mathbf{w}' f_0^-(\mathbf{w}) f_0^-(\mathbf{w}')\,\frac{\mathbf{g}^2\mathbf{I} - \mathbf{g}\,\mathbf{g}}{g^3}\right.\\ &\cdot\left[\nabla_w\Phi^-(\mathbf{w}) - \nabla_w\Phi^-(\mathbf{w})\right]\\ &\left.+ f_0^-(\mathbf{w})\,\frac{w^2\,\mathbf{I} - \mathbf{w}\,\mathbf{w}}{w^3}\cdot\nabla_w\Phi^-(\mathbf{w}),\right\}\end{aligned} \tag{57}$$

where

$$\mathbf{g} = \mathbf{w} - \mathbf{w}',\ \Lambda = \frac{e^2}{3\Theta}\left(\frac{4\pi N e^2}{\Theta}\right)^{1/2} \tag{58}$$

Equations (54)—(57) yield, correct to first order in $\Phi^{\pm}$, the following closed system of moment equations:

$$\frac{\partial\varrho}{\partial t} + \nabla\cdot(\varrho\,\mathbf{U}_0) = 0, \tag{59}$$

$$\begin{aligned}&\varrho\left(\frac{\partial}{\partial t} + \mathbf{U}_0\cdot\nabla\right)\mathbf{U}_0 + \nabla p + \varrho\nabla q\\ &\quad - \frac{1}{c}\mathbf{J}\times\mathbf{B} - \varepsilon\,\mathbf{E} = -\nabla\cdot(\mathbf{P} - p\mathbf{I}),\end{aligned} \tag{60}$$

and

$$\begin{aligned}&\left(\frac{\partial}{\partial t} + \mathbf{U}_0\cdot\nabla\right)p + \frac{5}{3}p\,\nabla\cdot\mathbf{U}_0 = -\frac{2}{3}(\mathbf{P} - p\mathbf{I}) : \nabla\mathbf{U}_0\\ &-\frac{2}{3}\nabla\cdot\mathbf{q} + \frac{2}{3}(\mathbf{J} - \varepsilon\,\mathbf{U}_0)\cdot\left(\mathbf{E} + \frac{1}{c}\,\mathbf{U}_0\times\mathbf{B}\right),\end{aligned} \tag{61}$$

where $p = (N_+ + N_-)\,\Theta = 2N\Theta$ is the isotropic part of the stress tensor. In the foregoing equations the quantities on the right hand side are determined in terms of Φ by the equations:

$$\mathbf{J} = \sum_{+,-} e\int d^3\mathbf{w}\,\mathbf{w} f_0^{\pm}\,\Phi^{\pm}, \tag{62}$$

$$\mathbf{P} - p\mathbf{I} = \sum_{+,-} m\int d^3\mathbf{w}\,\mathbf{w}\,\mathbf{w} f_0^{\pm}\,\Phi^{\pm} \tag{63}$$

and

$$\mathbf{q} = \sum \frac{m}{2}\int d^3\mathbf{w}\,\mathbf{w}\,w^2 f_0^{\pm}\,\Phi^{\pm}. \tag{64}$$

The most comprehensive calculations of transport co-efficients for a fully ionized gas are due to Marshall using effectively the model outlined here. We shall simply state his results. Introducing the auxiliary electric field

$$\mathbf{D} = \mathbf{E} + \frac{1}{c}\,\mathbf{U}_0\times\mathbf{B} + \frac{1}{2Ne}\nabla p, \tag{65}$$

and $\mathbf{n}$, the unit vector along the magnetic field, we obtain

$$\begin{aligned}\mathbf{J} - \varepsilon\,\mathbf{U}_0 &= \sigma_1\,\mathbf{n}\,\mathbf{n}\cdot\mathbf{D} + \sigma_2\,\mathbf{n}\times(\mathbf{D}\times\mathbf{n}) + \sigma_3\,\mathbf{n}\times\mathbf{D}\\ &+ \varkappa_1\,\mathbf{n}\,\mathbf{n}\cdot\nabla\Theta + \varkappa_2\,\mathbf{n}\times[(\nabla\Theta)\times\mathbf{n}] + \varkappa_3\,\mathbf{n}\times\nabla\Theta,\end{aligned} \tag{66}$$

where the various co-efficients are given by

$$\left.\begin{aligned}\sigma_1 &= 1.93\sigma_0\\ \sigma_2 &= a\sigma_0\,(\omega_e^2\tau^2 + 1.80)\\ \sigma_3 &= -a\sigma_0\,\omega_e\tau\,(\omega_e^2\tau^2 + 4.38)\end{aligned}\right\}, \tag{67}$$

$$\left.\begin{aligned} \varkappa_1 &= 1.55\,\varkappa_0 \\ \varkappa_2 &= -1.55\,a\varkappa_0\,(\omega_e^2\tau^2 - 0.97) \\ \varkappa_3 &= -4.30\,a\varkappa_0\,\omega_e\,\tau \end{aligned}\right\}, \tag{68}$$

and

$$\left.\begin{aligned} \sigma_0 &= \frac{N e^2 \tau}{m}, \qquad \varkappa_0 = \frac{N e \tau}{m} \\ a &= \frac{1}{\omega_e^4\tau^4 + 6.28\,\omega_e^2\tau^2 + 0.93} \end{aligned}\right\}. \tag{69}$$

Here ω_e denotes the electron cyclotron frequency and τ, the effective collision time, is given by

$$\tau = \frac{3}{(2\pi)^{\frac{1}{2}}}\left(\frac{m}{\Theta}\right)^{1/2}\frac{\Theta^2}{2 N e^2 \ln \Lambda} \tag{70}$$

where, it may be recalled, that $2N$ is the total number density.

The heat flow vector is given by

$$\begin{aligned} \mathbf{q} = &-\lambda_1\,\mathbf{n}\,\mathbf{n}\cdot\nabla\Theta - \lambda_2\,\mathbf{n}\times[(\nabla\Theta)\times\mathbf{n}] - \lambda_3\,\mathbf{n}\times\nabla\Theta \\ &-\nu_1\,\mathbf{n}\,\mathbf{n}\cdot\mathbf{D} - \nu_2\,\mathbf{n}\times(\mathbf{D}\times\mathbf{n}) - \nu_3\,\mathbf{n}\times\mathbf{D}, \end{aligned} \tag{71}$$

where

$$\left.\begin{aligned} \lambda_1 &= 3.59\,\lambda_0 \\ \lambda_2 &= \lambda_0\,[a\,(.458\,\omega_e^2\tau^2 + 3.01) + 4.46\,b] \\ \lambda_3 &= 1.25\,\lambda_0\,[b\,\omega_e\,\tau - a\,\omega_e\,\tau\,(\omega_e^2\tau^2 + 6.2)] \end{aligned}\right\}, \tag{72}$$

$$\left.\begin{aligned} \nu_1 &= 3.19\,\nu_0 \\ \nu_2 &= a\nu_0\,(0.5\,\omega_e^2\tau^2 + 2.98) \\ \nu_3 &= -a\,\nu_0\,(1.25\,\omega_e^2\tau^2 + 7.63) \end{aligned}\right\}, \tag{73}$$

nd

$$\left.\begin{aligned} \lambda_0 &= \frac{2N\Theta\tau}{m}, \qquad \nu_0 = \frac{2Ne\Theta\tau}{m} \\ b &= \frac{1}{\omega_e^2\tau^2 + 12.72} \end{aligned}\right\}, \tag{74}$$

It is clear from Eqs. (66) and (71) that, in the presence f a temperature gradient, not only will there appear a ow of heat **q**, but also, a flow of electric current. Similrly, an electric field produces a flow of heat. According the thermodynamics of irreversible processes, these our co-efficients are not independent as three of them re linked by the Onsager relations

$$\nu_i = \Theta\left(\varkappa_i + \frac{5}{2}\frac{\sigma_i}{e}\right), \tag{75}$$

here *i* takes the value 1, 2, or 3.

In a representation in which the magnetic field is ong the *z*-axis, Eqs. (66) and (71) take the form

$$\mathbf{J} - \varepsilon\,\mathbf{U}_0 = \boldsymbol{\sigma}\cdot\mathbf{D} + \boldsymbol{\varkappa}\cdot\nabla\Theta \tag{76}$$

d

$$\mathbf{q} = -\boldsymbol{\lambda}\cdot\nabla\Theta - \boldsymbol{\nu}\cdot\mathbf{D}, \tag{77}$$

here

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_2 & -\sigma_3 & 0 \\ \sigma_3 & \sigma_2 & 0 \\ 0 & 0 & \sigma_1 \end{pmatrix}, \tag{78}$$

d the dyadics $\boldsymbol{\varkappa}$, $\boldsymbol{\lambda}$, and $\boldsymbol{\nu}$ also have similar structure.

A particularly interesting situation results when, in the equilibrium state, $\varepsilon = 0$, and

$$J_1 = 0, \qquad \nabla p = 0, \qquad \nabla\Theta = 0, \quad \text{and } D_3 = 0. \tag{79}$$

Eq. (78) then yields

$$D_1 = \frac{\sigma_3}{\sigma_2}\,D_2 \tag{80}$$

and

$$J_2 = \sigma_\perp\,D_2, \tag{81}$$

where

$$\sigma_\perp = \frac{\sigma^2{}_2 + \sigma^2{}_3}{\sigma_2}, \tag{82}$$

$$\simeq \sigma_1 \qquad \text{when } \omega_e\,\tau \ll 1. \tag{83}$$

$$\simeq \sigma_1 / 1.93 \quad \text{when } \omega_e\,\tau \gg 1. \tag{84}$$

In presenting the expressions for the components of the stress tensor, it is convenient to introduce the coefficient of viscosity, in the absence of a magnetic field,

$$\mu = \frac{1}{3}N\Theta\,\tau_2, \qquad \text{where } \tau_2 = \frac{5}{3\,(2)^{1/2}}\left(\frac{M}{m}\right)^{1/2}\tau, \tag{85}$$

and the symmetric traceless tensor

$$\mathbf{A} = \frac{1}{2}\left[\nabla\,\mathbf{U}_0 + (\nabla\,\mathbf{U}_0)^T\right] - \frac{1}{3}\nabla\cdot\mathbf{U}_0\,\mathbf{I}. \tag{86}$$

Then, in a co-ordinate system in which **B** is along the *z*-axis, we have

$$\left.\begin{aligned} P_{zz} &= p - 2\mu\,A_{zz}, \\ P_{yy} &= p - 2\mu\,\frac{1}{1 + 4\,\omega_i^2\tau_2^2} \\ &\quad [A_{yy} + 2\omega_i^2\tau_2^2\,(A_{xx} + A_{yy}) + 2\,\omega_i\,\tau_2 A_{yx}], \\ P_{xx} &= p - 2\mu\,\frac{1}{1 + 4\,\omega_i^2\tau_2^2} \\ &\quad [A_{xx} + 2\,\omega_i^2\tau_2^2\,(A_{xx} + A_{yy}) - 2\,\omega_i\,\tau_2 A_{yx}], \\ P_{xz} &= P_{zx} = -\frac{2\mu}{1 + \omega_i^2\tau_2^2}\,(A_{xz} + \omega_i\,\tau_2\,A_{yz}) \\ P_{yz} &= P_{zy} = -\frac{2\mu}{1 + \omega_i^2\tau_2^2}\,(A_{yz} - \omega_i\,\tau_2\,A_{xz}), \\ P_{xy} &= P_{yx} = \\ &\quad -\frac{2\mu}{1 + \omega_i^2\tau_2^2}\left[A_{xy} - \omega_i\,\tau_2\,(A_{xx} - A_{yy})\right]. \end{aligned}\right\} \tag{87}$$

where ω_i denotes the ion-cyclotron frequency.

2.1 *Weak Magnetic Fields*

In the limiting case of a weak magnetic field $\omega_{e,i}\,\tau \ll 1$, and it is to be observed from the foregoing relations that in this limit, to the lowest significant order,

$$\begin{aligned} &\sigma_1 \simeq \sigma_2, \quad \sigma_3 \simeq 0, \quad \varkappa_1 \simeq \varkappa_2, \quad \varkappa_3 \simeq 0, \\ &\lambda_1 \simeq \lambda_2, \quad \lambda_3 \simeq 0, \quad \nu_1 \simeq \nu_2, \quad \nu_3 \simeq 0. \end{aligned} \tag{88}$$

Expressions (76), (77), and (87) now take the simple form

$$\mathbf{J} - \varepsilon\,\mathbf{U}_0 = \sigma_1\,\mathbf{D} + \varkappa_1\,\nabla\Theta, \tag{89}$$

$$\mathbf{q} = -\lambda_1\,\nabla\Theta - \nu_1\,\mathbf{D}, \tag{90}$$

and

$$\mathbf{P} = p\,\mathbf{I} + \frac{2}{3}\mu\nabla\cdot\mathbf{U}_0\mathbf{I} - \mu\left[\nabla\mathbf{U}_0 + (\nabla\mathbf{U}_0)^T\right]. \tag{91}$$

On substituting Eqs. (89)—(91) into the equation of state (61), we obtain

$$\begin{aligned}\varrho^{\gamma}\frac{d}{dt}(p\varrho^{-\gamma}) &= \frac{2}{3}\nabla\cdot(\lambda_1\nabla\Theta + \nu_1\mathbf{D})\\ &+ \frac{4\mu}{3}\mathbf{A}:\mathbf{A} + \frac{2}{3\sigma_1}\left|\mathbf{J} - \varepsilon\mathbf{U}_0\right|^2\\ &- \left(\frac{1}{3Ne}\nabla p + \frac{2}{3}\varkappa_1\nabla\Theta\right)\cdot\left(\mathbf{E} + \frac{1}{c}\mathbf{U}_0\times\mathbf{B}\right),\end{aligned} \tag{92}$$

where $\gamma = 5/3$. From the Maxwell equation $c\nabla \times \mathbf{E} = -\partial\mathbf{B}/\partial t$ we obtain

$$\frac{\partial\mathbf{B}}{\partial t} = \nabla\times(\mathbf{U}_0\times\mathbf{B}) - \frac{c}{\sigma_1}\nabla\times(\mathbf{J} - \varepsilon\mathbf{U}_0). \tag{93}$$

Thus our complete set of hydromagnetic equations are:

$$\frac{\partial\varrho}{\partial t} + \nabla\cdot(\varrho\mathbf{U}_0) = 0, \tag{94}$$

$$\begin{aligned}\varrho\left(\frac{\partial}{\partial t} + \mathbf{U}_0\cdot\nabla\right)\mathbf{U}_0 &= -\nabla p + \varepsilon\mathbf{E} + \frac{1}{c}\mathbf{J}\times\mathbf{B} - \varrho\nabla\varphi\\ &+ \mu\nabla^2\mathbf{U}_0 + \frac{1}{3}\mu\nabla\nabla\cdot\mathbf{U}_0,\end{aligned} \tag{95}$$

$$\frac{\partial\mathbf{B}}{\partial t} = \nabla\times(\mathbf{U}_0\times\mathbf{B}) - \frac{c}{\sigma_1}\nabla\times(\mathbf{J} - \varepsilon\mathbf{U}_0), \tag{96}$$

$$\begin{aligned}\varrho^{\gamma}\left(\frac{\partial}{\partial t} + \mathbf{U}_0\cdot\nabla\right)(p\varrho^{-\gamma}) &= \frac{2}{3}\nabla\cdot(\lambda_1\nabla\Theta + \nu_1\mathbf{D})\\ &+ \frac{4}{3}\mu\mathbf{A}:\mathbf{A} + \frac{2}{3\sigma_1}\left|\mathbf{J} - \varepsilon\mathbf{U}_0\right|^2\\ &- \left(\frac{1}{3Ne}\nabla p + \frac{2}{3}\varkappa_1\nabla\Theta\right)\cdot\left(\mathbf{E} + \frac{1}{c}\mathbf{U}_0\times\mathbf{B}\right),\end{aligned} \tag{97}$$

where the charge and current densities are determined by the Maxwell equations. For an ideal, perfectly conducting fluid, $\sigma_1, \to \infty$,

$$\mathbf{E} + \frac{1}{c}\mathbf{U}_0\times\mathbf{B} = 0,\ \mu \to 0,\ \mathbf{q} = 0,$$

and if the characteristic frequencies of interest are much less than the ion cyclotron frequency and the foregoing relations reduce to the well-known hydromagnetic equations:

$$\frac{\partial\varrho}{\partial t} + \nabla\cdot(\varrho\mathbf{U}) = 0, \tag{98}$$

$$\begin{aligned}\varrho\left(\frac{\partial}{\partial t} + \mathbf{U}\cdot\nabla\right)\mathbf{U} &=\\ -\nabla p &+ \frac{1}{c}(\mathbf{J} - \varepsilon\mathbf{U})\times\mathbf{B} - \varrho\nabla\varphi,\end{aligned} \tag{99}$$

$$\frac{\partial\mathbf{B}}{\partial t} = \nabla\times(\mathbf{U}\times\mathbf{B}), \tag{100}$$

and

$$\frac{d}{dt}\left(p\varrho^{-\gamma}\right) = 0, \tag{101}$$

where in writing these equations we have dropped the subscript on $\mathbf{U}_0$, as it is no longer necessary.

2.2 The A. C. Conductivity

There is one further important transport co-efficient which has not yet been worked out for a plasma. Since one is often interested in frequencies of the electromagnetic fields comparable to the effective collision frequency, the usual Chapman-Enskog theory needs some modification. This can be effected readily by introducing terms in $\partial\Phi^{\pm}/\partial t$ on the left hand side of Eqs. (54) and (55). Now it is the electrons which primarily determine the conductivity. For the purpose of an order-of-magnitude estimate, we can neglect electron-electron collisions. Then, in the absence of temperature and density gradients, magnetic fields, or mass flows, Eq. (55) reduces to

$$\begin{aligned}f_0^-(\mathbf{w})\left(\frac{\partial\Phi^-}{\partial t} + \frac{e}{\Theta}\mathbf{E}\cdot\mathbf{w}\right) &=\\ \left(\frac{\Theta}{m}\right)^{3/2}\frac{1}{\tau_1}\frac{\partial}{\partial\mathbf{w}}&\cdot\left[\frac{w^2\mathbf{I} - \mathbf{w}\mathbf{w}}{w^3}\cdot\frac{\partial\Phi^-}{\partial\mathbf{w}}f_0^-(w)\right],\end{aligned} \tag{102}$$

where

$$\tau_1 = \left(\frac{m}{\Theta}\right)^{1/2}\frac{\Theta^2}{2\pi Ne^4\ln\Lambda}. \tag{103}$$

If $\mathbf{E}$ varies in time like $\exp(-i\omega t)$, then so must Φ^-, and the solution of Eq. (102) is

$$\Phi^- = -e\frac{\mathbf{E}\cdot\mathbf{w}}{2\Theta}\tau_1\frac{[(mw)^2/2\Theta]^{3/2}}{1 + \frac{1}{2}i\omega\tau_1(mw^2/\Theta)^{3/2}}. \tag{104}$$

The associated current is then given by

$$\mathbf{J} = -e\int d^3\mathbf{w}\,\mathbf{w}\,f_0^-(\mathbf{w})\,\Phi^-, \tag{105}$$

which, on using $(\Theta/m)^{1/2}$ as the unit of velocity, can be written in the form

$$\mathbf{J} = \sigma\mathbf{E}, \tag{106}$$

where

$$\sigma = \frac{Ne^2\tau_1}{m}\frac{1}{(2\pi)^{1/2}}\frac{1}{3}\int_0^{\infty}\frac{dw\,w^7e^{-w^2/2}}{1 + \frac{1}{2}i\omega\tau_1w^3}. \tag{107}$$

If $\omega\tau_1 \ll 1$, Eq. (107) yields to the lowest significant order

$$\sigma = \frac{Ne^2\tau_1}{m}\frac{1}{(2\pi)^{1/2}}\frac{1}{3}\left(32 - \frac{945}{4}i\omega\tau_1\right); \tag{108}$$

whereas if $\omega\tau_1 \gg 1$, we obtain

$$\sigma \simeq -\frac{iNe\tau_1}{m}\frac{1}{\omega\tau_1}\left(1 + i\frac{2}{3(2\pi)^{1/2}}\frac{1}{\omega\tau_1}\right). \tag{109}$$

3. The Case of Weak Collisions

If the plasma is so dilute, or the temperature so high that the direct interparticle collisions of the first and second class described earlier are negligible, then the particles interact mainly via the macroscopic electromagnetic fields. We may thus drop altogether the collision term in the Boltzmann equation (3) in treating the dynamics of the plasma. Firstly, we wish to emphasize that this description of a plasma via the collisionless Boltzmann equation is entirely equivalent to the solution of the equations of motion for the particle orbits. This can be shown as follows.

3.1 Equivalence of Collisionless Bolzmann Equation and Particle Orbit Theory

Consider the collisionless Boltzmann equation or what is referred to as the kinetic equation in the Russian literature:

$$\frac{\partial f}{\partial t} + (\mathbf{v} \cdot \nabla) f + \mathbf{F} \cdot \nabla_v f = 0 . \tag{110}$$

The Lagrangian subsidiary equations are:

$$\mathbf{F} = \frac{d\mathbf{v}}{dt}, \tag{111}$$

and

$$\mathbf{v} = \frac{d\mathbf{x}}{dt}. \tag{112}$$

Let the solutions to these equations be

$$\mathbf{v} = \mathbf{v}(\alpha_1, \alpha_2 \cdots \alpha_6, t) \tag{113}$$

and

$$\mathbf{x} = \mathbf{x}(\alpha_1, \alpha_2, \cdots \alpha_6, t), \tag{114}$$

where the α's are the integration constants. We further uppose that Eqs. (113) and (114) can be solved for the 's. Now the general solution of Eq. (110) is an arbitrary unction of the constants of motion, i.e.,

$$f(\mathbf{x}, \mathbf{v}, t) = \mathrm{f}(\alpha_1, \alpha_2, \cdots \alpha_6); \tag{115}$$

or substituting Eq. (115) into (110) we obtain

$$\sum_{i=1}^{6} \frac{\partial f}{\partial \alpha_i} \left(\frac{\partial \alpha_i}{\partial t} + \mathbf{v} \cdot \nabla \alpha_i + \mathbf{F} \cdot \nabla_v \alpha_i \right)$$

$$= \sum_{i=1}^{6} \frac{\partial t}{\partial \alpha_i} \frac{d\alpha_i}{dt} = 0 \tag{116}$$

nce the α's are constants along a particle trajectory. Thus we have shown that the most general solution of q. (110) is an arbitrary function of the integrals of the agrangian subsidiary equations (111) and (112). But hese are just the equations of motion for the particle rbits. The equivalence of the particle orbit theory and he collisionless Boltzmann equation was first demonrated by Jeans, and in the astronomical literature is ferred to as Jeans' theorem. When the α's are not true onstants but adiabatic invariants, the resulting f saties the Boltzmann equation to within the accuracy of hese invariants. The considerations of the orbit theory rovide a better insight into the physical factors involved a particular problem. However, the Boltzmann quation provides a rigorous treatment of complex prolems which may not be easily accessible to the orbit eory.

3.2 The Low Temperature Approximation

Neglecting, then, the right hand side of Eq. (49), the equation for the stress tensor now becomes

$$\frac{\partial}{\partial t} \mathbf{P} + \nabla \cdot (\mathbf{Q} + \mathbf{U}\,\mathbf{P}) + \mathbf{P} \cdot \nabla \mathbf{U} + (\mathbf{P} \cdot \nabla \mathbf{U})^r$$
$$+ \frac{e}{mc} (\mathbf{B} \times \mathbf{P} - \mathbf{P} \times \mathbf{B}) - (\mathbf{J} - \varepsilon \mathbf{U}) \left(\mathbf{E} + \frac{1}{c} \mathbf{U} \times \mathbf{B} \right)$$
$$- \left(\mathbf{E} + \frac{1}{c} \mathbf{U} \times \mathbf{B} \right) (\mathbf{J} - \varepsilon \mathbf{U}) = 0 . \tag{117}$$

In order of magnitude $|\partial \mathbf{P}/\partial t| \sim \omega v_{th}^3$ and $|\nabla \cdot \mathbf{Q}| \sim k v_{th}^4$ where $v_{th} = (\Theta/m)^{1/2}$, and ω and k denote the characteristic frequency and reciprocal length respectively. Therefore, it is clear that if

$$\frac{\omega}{k} \gg v_{th}, \tag{118}$$

the heat flow tensor can be neglected in Eq. (117). For the linearized equations of motion ω/k is the phase velocity of the wave, while for a non-linear system it might be termed the effective phase velocity. Thus if the phase velocity of the wave is much larger than the thermal speed of the ions or electrons, Eq. (117) reduces to

$$\frac{\partial}{\partial t} \mathbf{P} + \nabla \cdot (\mathbf{U}\,\mathbf{P}) + \mathbf{P} \cdot \nabla \mathbf{U} + (\mathbf{P} \cdot \nabla \mathbf{U})^r$$
$$+ \frac{e}{mc} (\mathbf{B} \times \mathbf{P} - \mathbf{P} \times \mathbf{B}) - (\mathbf{J} - \varepsilon \mathbf{U}) \left(\mathbf{E} + \frac{1}{c} \mathbf{U} \times \mathbf{B} \right)$$
$$- \left(\mathbf{E} + \frac{1}{c} \mathbf{U} \times \mathbf{B} \right) (\mathbf{J} - \varepsilon \mathbf{U}) = 0 . \tag{119}$$

Thus, by virtue of the approximation (118) we are able to obtain a closed set of moment equations. This scheme will be referred to as the *Low Temperature Approximation* and will yield results correct to first significant order in the parameter of smallness $k v_{th}/\omega$.

Clearly, one can decouple the nth moment equation from its dependence on the $(n+1)$th moment in an analogous fashion, and obtain the corresponding higher order thermal corrections. In all such calculations care must be taken to discard any spurious roots which may occur, that is, roots which are incompatible with the basic assumption (118).

3.3 Chew, Goldberger and Low Approximation

If we now assume that the magnetic field is so strong that the electron or ion cyclotron frequency is very large compared to the characteristic frequencies of interest, we can make a systematic expansion of the Boltzmann equation (110) in inverse powers of B. This has been carried out by Chew, Goldberger, and Low. An equivalent procedure is to carry out this expansion in the equation of motion (119) for the stress tensor. If we assume that the Joule dissipation is negligible, i.e.,

$$\left| \left(\mathbf{E} + \frac{1}{c} \mathbf{U} \times \mathbf{B} \right) (\mathbf{J} - \varepsilon \mathbf{U}) \right| \ll \left| \mathbf{B} \times \mathbf{P} \right|, \tag{120}$$

then the leading term in Eq. (119) gives

$$\mathbf{B} \times \mathbf{P} = \mathbf{P} \times \mathbf{B} . \tag{121}$$

It immediately follows from this equation that

$$\mathbf{P} = P_\perp \mathbf{I} + (P_\parallel - P_\perp)\,\mathbf{n}\,\mathbf{n}, \tag{122}$$

where $\mathbf{n}$ denotes a unit vector along $\mathbf{B}$. Eq. (122) implies that the pressure tensor is diagonal in a local rectangular co-ordinate system, one of whose axes points along $\mathbf{n}$. In the plane perpendicular to $\mathbf{n}$ the pressure is a scalar of magnitude $P_\perp$, and the pressure along $\mathbf{B}$ is $P_\parallel$ which, in general, is not equal to $P_\perp$. Equation (119) now reduces to

$$\begin{aligned} &\frac{d}{dt}[P_\perp \mathbf{I} + (P_\parallel - P_\perp)\,\mathbf{n}\,\mathbf{n}] + [P_\perp \mathbf{I} + \\ &(P_\parallel - P_\perp)\mathbf{n}\mathbf{n}]\,\nabla\cdot\mathbf{U} + [P_\perp \mathbf{I} + (P_\parallel - P_\perp)\mathbf{n}\mathbf{n}]\cdot\nabla\mathbf{U} \\ &+ \{[P_\perp \mathbf{I} + (P_\parallel - P_\perp)\,\mathbf{n}\,\mathbf{n}]\cdot\nabla\mathbf{U}\}^T = 0, \end{aligned} \tag{123}$$

where we have now set

$$\mathbf{E} + \frac{1}{c}\mathbf{U}\times\mathbf{B} = 0, \tag{124}$$

Eq. (124) implies that the medium has very high electrical conductivity so that it is unable to support, in its rest frame, any significant electric fields. Taking the trace of Eq. (123), we obtain

$$\begin{aligned} &\frac{d}{dt}\left(P_\perp + \frac{1}{2}P_\parallel\right) + \left(P_\perp + \frac{1}{2}P_\parallel\right)\nabla\cdot\mathbf{U} + P_\perp\nabla\cdot\mathbf{U} \\ &+ (P_\parallel - P_\perp)\,\mathbf{n}\,\mathbf{n}:\nabla\mathbf{U} = 0. \end{aligned} \tag{125}$$

The **nn** component of Eq. (123) is

$$\frac{d}{dt}P_\parallel + P_\parallel\nabla\cdot\mathbf{U} + 2P_\parallel\mathbf{n}\,\mathbf{n}:\nabla\mathbf{U} = 0. \tag{126}$$

From Eqs. (125) and (126) we readily obtain

$$\frac{d}{dt}P_\perp + 2P_\perp\nabla\cdot\mathbf{U} - P_\perp\mathbf{n}\,\mathbf{n}:\nabla\mathbf{U} = 0, \tag{127}$$

Using the relation (124) in the Maxwell equation $c\nabla\times\mathbf{E} = -\partial\mathbf{B}/\partial t$, we obtain

$$\frac{\partial\mathbf{B}}{\partial t} = \nabla\times(\mathbf{U}\times\mathbf{B}). \tag{128}$$

It is convenient to write Eq. (128) as

$$\frac{d\mathbf{B}}{dt} = \left(\frac{\partial}{\partial t} + \mathbf{U}\cdot\nabla\right)\mathbf{B} = (\mathbf{B}\cdot\nabla)\,\mathbf{U} - \mathbf{B}\,(\nabla\cdot\mathbf{U}). \tag{129}$$

From Eq. (129) and the equation of continuity

$$\frac{d\varrho}{dt} + \varrho\nabla\cdot\mathbf{U} = 0, \tag{130}$$

it can be shown readily that

$$\mathbf{n}\,\mathbf{n}:\nabla\mathbf{U} = \frac{\varrho}{B}\frac{d}{dt}\left(\frac{B}{\varrho}\right) \tag{131}$$

where B denotes the magnitude of $\mathbf{B}$. On using this result in Eqs. (126) and (127), we obtain the two adiabatic relations

$$\frac{d}{dt}\left(\frac{P_\parallel B^2}{\varrho^3}\right) = 0 \tag{132}$$

and

$$\frac{d}{dt}\left(\frac{P_\perp}{\varrho B}\right) = 0. \tag{133}$$

Starting from the Boltzmann equation (110), Chew, Goldberger, and Low have derived the same adiabatic relations on the assumption that the heat transport along the lines of force is negligible.

Equation (133) implies that, if one moves along with an element of the fluid, the average transverse kinetic energy per particle is proportional to B. This is compatible with the well known constancy of the magnetic moment of a charged particle in the adiabatic approximation. It can be shown that Eq. (132) implies the constancy of the longitudinal invariant of Chew, Goldberger, and Low. Thus for a low density plasma in the presence of a strong magnetic field, in the framework of the low temperature approximation or on the assumption of negligible heat transport along the lines of force and when the conductivity is very high, we obtain the following complete system of equations:

$$\frac{\partial\varrho}{\partial t} + \nabla\cdot(\varrho\,\mathbf{U}) = 0, \tag{135}$$

$$\begin{aligned} \varrho\frac{d\mathbf{U}}{dt} &= \varrho\left(\frac{\partial}{\partial t} + \mathbf{U}\cdot\nabla\right)\mathbf{U} \\ &\quad - \nabla\cdot\mathbf{P} + \frac{1}{c}(\mathbf{J} - \varepsilon\,\mathbf{U})\times\mathbf{B} - \varrho\,\nabla\varphi, \end{aligned} \tag{136}$$

$$\frac{\partial\mathbf{B}}{\partial t} = \nabla\times(\mathbf{U}\times\mathbf{B}), \tag{137}$$

$$\mathbf{P} = P_\perp\mathbf{I} + (P_\parallel - P_\perp)\,\mathbf{n}\,\mathbf{n}, \tag{138}$$

$$\frac{d}{dt}\left(\frac{P_\parallel B^2}{\varrho^3}\right) = 0, \tag{139}$$

and

$$\frac{d}{dt}\left(\frac{P_\perp}{\varrho B}\right) = 0, \tag{140}$$

where the charge and current densities are determined by the Maxwell equations.

III. Wave Phenomena in Plasmas

1. General Remarks

We shall now discuss the behaviour of plasmas for states initially close to a steady state. This implies the existence of a time-independent, self-consistent solution of the appropriate equations of motion, jointly with the Maxwell equations, subject to proper boundary conditions. The notion of closeness implies the existence of some dimensionless parameter of smallness, in terms of which one can expand the time dependent quantities of the equations governing the system. In most situation the appropriate parameter is the ratio of the macroscopic velocity to the phase velocity. The lowest order system of the expanded equations is just that governing the steady state. To next order one obtains a set of linear equations, whose coefficients do not in general depend on time, which describe the small motions about the steady state.

If there exists no small initial perturbation whose amplitude grows monotonically or sinusoidally in time then the system is said to be *stable*; otherwise, *unstable*. In the absence of any dissipative mechanisms the small motions of a stable system may tend to persist in definitely; whereas in the presence of a dissipative mechanism they may tend to die out, the system returning to the steady state.

If the system is unstable, after a certain interval of time the amplitude of the motion will become so large that the linearized equations will cease to be valid. The subsequent behaviour of the system will be governed by non-linear equations, whose analysis, in general, is difficult to carry out.

There are two plausible situations into which these instabilities might possibly carry the system. The first of these might be termed a turbulent state, characterized by randomly fluctuating electromagnetic fields and motions, which could manifest itself by an enhancement of diffusion and resistivity over and above that resulting from collisions. The second situation is associated with instabilities whose exponentiation time is short, can affect significantly the heavy particles and may lead to coherent macroscopic motion of the plasma, resulting in a radically different final state. The former of these has been conjectured as a cause of the loss of containment in the stellarator type of controlled fusion devices. An example of the latter has been observed in pinch effect experiments where the discharge has destroyed itself upon achievement of a critical current.

Since the coefficients of the linearized equations do not depend explicitly on time, one can seek a complete set of characteristic solutions whose time dependence is of the form

$$\exp(-i\,\omega\,t)\,. \tag{1}$$

This leads to a characteristic value problem for ω which, in general, may be complex. If there exists an ω for which

$$\operatorname{Im}\omega > 0\,, \tag{2}$$

hen the system is unstable. If the coefficients also appen to be independent of one or more of the space variables, one can seek solutions of the form $\exp(ikx)$. The solution of the characteristic value problem, subect to the appropriate boundary conditions, then leads o a relation between ω and k which may be written mplicitly in the form

$$F(\omega, k) = 0\,. \tag{3}$$

A relation of the form (3) is usually referred to as the *ispersion relation.* The solution of Eq. (3) for ω in terms f k would immediately disclose whether the system is table or unstable.

Sometimes it is convenient to characterize the family f steady states by a certain parameter, say β. As one hanges β continuously, the system may pass from tability to instability for some determinate value of β, ay $\beta = \beta_c$; i.e., ω passes from the real axis into the omplex ω plane (with $\operatorname{Im}\omega > 0$) at $\beta = \beta_c$. The state, haracterized by β_c, which marks the boundary between he stable and the unstable regions is referred to as the *arginal state.*

In the case of static equilibria for non-dissipative ystems governed by the hydromagnetic equations, it as been shown by Bernstein, Frieman, Kruskal, and ulsrud that the linearized equations of motion are selfadjoint; consequently the characteristic values ω^2 are real. Therefore,

$$\omega^2 > 0 \tag{4a}$$

corresponds to stability and

$$\omega^2 < 0\,, \tag{4b}$$

to instability. However, for more general types of equilibria, the characteristic values ω may be complex. In case the imaginary part of ω is positive, the system will be unstable; following Eddington, this type of instability is referred to as *overstability.*

In addition to questions of stability which are usually posed for isolated systems, there is the important class of driven problems. In these situations a system in a steady state is excited by an external agency, perhaps at a fixed frequency. When the amplitude of excitation is small, the problem can again be treated by the same set of linear equations as those employed for the analysis of stability, but with appropriate modification of the boundary conditions. In the cases where a Fourier analysis in space is in order, one is again led to a dispersion relation, but now it is to be viewed as determining the wave number k (in general complex) as a function of the prescribed frequency ω, assumed real. The fact that k can be complex permits waves to grow or die out in space, giving rise to the phenomena of amplification and evanescence. Problems of reflection, transmission, scattering, and amplification of waves by plasmas fall into this category. It must be emphasized that when any particular problem is properly posed with due regard to boundary conditions, there will be no confusion about the interpretation of the resultant dispersion relation.

2. UNBOUNDED PLASMAS IN THE ABSENCE OF EXTERNAL FIELDS

2.1 *Longitudinal Electron Plasma Oscillation*

We first consider the simplest case of electron oscillations in a uniform plasma, neglecting the thermal motion of the electrons. The ions will be assumed to be at rest and to form the uniform neutralizing background for the electron charge in equilibrium. Let N denote the electron density (which is equal to the ion density) in equilibrium. In the perturbed state let the density be denoted by $N + n$, where n is considered small so that all terms quadratic or higher in n can be neglected. The equations of motion now are (cf. Eqs. (II-44) and (II-45)):

$$\frac{\partial n}{\partial t} + N\nabla\cdot\mathbf{v} = 0\,, \tag{5}$$

and

$$m\frac{\partial \mathbf{v}}{\partial t} = -e\,\mathbf{E}\,, \tag{6}$$

where $-e$ and m denote the charge and mass of the electron respectively. For longitudinal oscillations the electric field is determined simply by the Poisson equation

$$\nabla\cdot\mathbf{E} = 4\pi\varepsilon = -4\pi e n\,. \tag{7}$$

Eliminating **v** and **E** from Eqs. (5)—(7) we obtain for the equation of motion

$$\frac{\partial^2 n}{\partial t^2} = -\frac{4\pi N e^2 n}{m} = -\omega_p{}^2 n, \tag{8}$$

where

$$\omega_p = \left(\frac{4\pi N e^2}{m}\right)^{1/2} = 2\pi \times 8980\, N^{1/2}\,\mathrm{sec}^{-1}. \tag{9}$$

It follows from Eq. (8) that the density fluctuations oscillate sinusoidally with the characteristic frequency ω_p. The frequency ω_p is referred to as the *plasma frequency*. Formula (9) was first obtained by Tonks and Langmuir in 1929.

The absence of space co-ordinates in Eq. (8) shows that these waves are non-dispersive—their group velocity vanishes — and there is no tendency for a wave packet of this type to propagate through the plasma. As a result we can specify the phases of the electron displacement in such a way as to obtain a traveling wave —a wave, however, which moves continuously through a fixed region without ever progressing beyond.

2.2 *Transverse Electron Plasma Oscillations*

We shall now consider the transverse (or electromagnetic) plasma oscillations, neglecting the thermal motions of the electrons. The equations governing the problem are the linearized equations of motion (cf. Eqs. (II-44) and (II-45)) and the Maxwell equations

$$\frac{\partial n}{\partial t} + N\nabla\cdot\mathbf{v} = 0, \tag{10}$$

$$m\frac{\partial \mathbf{v}}{\partial t} = -e\mathbf{E}, \tag{11}$$

$$\nabla\times\mathbf{B} = -\frac{4\pi N e}{c}\mathbf{v} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}, \tag{12}$$

and

$$\nabla\times\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}. \tag{13}$$

From Eqs. (10)—(13) we readily obtain

$$\frac{\partial^2 \mathbf{E}}{\partial t^2} = -\omega_p{}^2\mathbf{E} - c^2\nabla\times\nabla\times\mathbf{E}. \tag{14}$$

Taking **E** to be of the form $\exp i(-\omega t + \mathbf{k}\cdot\mathbf{r})$, Eq. (14) leads to

$$(\omega^2 - \omega_p{}^2 - c^2k^2)\,\mathbf{k}\times\mathbf{E} = 0, \tag{15}$$

and

$$(\omega^2 - \omega_p{}^2)\,(\mathbf{k}\cdot\mathbf{E}) = 0. \tag{16}$$

For transverse oscillations $\mathbf{k}\times\mathbf{E}$ does not vanish and Eq. (15) gives the dispersion relation

$$\omega^2 = \omega_p{}^2 + c^2k^2; \tag{17}$$

while for longitudinal oscillations $\mathbf{k}\cdot\mathbf{E}$ does not vanish and Eq. (16) gives the dispersion relation already discussed. In contrast to the longitudinal oscillations, the transverse oscillations propagate in the plasma with the group velocity.

$$\mathbf{v}_g = \frac{\partial \omega}{\partial \mathbf{k}} = \frac{c\,\mathbf{k}}{k\,[1 + (\omega_p{}^2/c^2k^2)]^{1/2}} \tag{18}$$

2.3 *Effect of Thermal Motions on Electron Plasma Oscillations*

We shall now treat electron plasma oscillations, taking into account the thermal motions of the electrons using the low temperature approximation developed in Section II. In the equilibrium state, we shall assume the plasma pressure to be a scalar; however, we do allow for anisotropy of the material stress tensor during the perturbations. The perturbed state is characterized by

$$\mathbf{v}, \quad N + n, \quad p\mathbf{I} + \mathsf{P}_1, \tag{19}$$

where $p = N\Theta$ denotes the electron pressure in equilibrium, and P_1, the perturbation in the stress tensor. The linearized equations of motion now are (cf. Eqs. (II-44) to (II-)46):

$$\frac{\partial n}{\partial t} + N\nabla\cdot\mathbf{v} = 0, \tag{20}$$

$$mN\frac{\partial \mathbf{v}}{\partial \mathrm{t}} = -\nabla\cdot\mathsf{P}_1 - Ne\mathbf{E}, \tag{21}$$

and

$$\frac{\partial}{\partial t}\mathsf{P}_1 = -p\,(\nabla\cdot\mathbf{v})\,\mathbf{I} - p\,[\nabla\mathbf{v} + (\nabla\mathbf{v})^T]. \tag{22}$$

Equations (20)—(22) must be solved in conjunction with the linearized Maxwell equations

$$\nabla\times\mathbf{B} = -\frac{4\pi N e}{c}\mathbf{v} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} \tag{23}$$

and

$$\nabla\times\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}. \tag{24}$$

Introducing $\boldsymbol{\xi}$, the Lagrangian displacement of an element of the electron fluid from its equilibrium position, to a first approximation, we can write

$$\mathbf{v} = \frac{\partial \boldsymbol{\xi}}{\partial t}. \tag{25}$$

Eqs. (20) and (22) can then be readily integrated to give

$$n = -N\nabla\cdot\boldsymbol{\xi} \tag{26}$$

and

$$\mathsf{P}_1 = -p\,(\nabla\cdot\boldsymbol{\xi})\,\mathbf{I} - p\,[\nabla\boldsymbol{\xi} + (\nabla\boldsymbol{\xi})^T], \tag{27}$$

where the constants of integration have, by definition, been set equal to zero. Assuming the space time behaviour of all quantities to be of the form $\exp i(-\omega t + \mathbf{k}\cdot\mathbf{r})$, the foregoing equations yield

$$\omega^2\boldsymbol{\xi} = 2\frac{\Theta}{m}\mathbf{k}\,\mathbf{k}\cdot\boldsymbol{\xi} + \frac{\Theta}{m}k^2\boldsymbol{\xi} - \frac{e}{m}\mathbf{E}, \tag{28}$$

and

$$c^2\mathbf{k}\times\mathbf{k}\times\mathbf{E} = +4\pi Ne^2\omega^2\boldsymbol{\xi} - \omega^2\mathbf{E}, \tag{29}$$

From Eqs. (28) and (29) we obtain for longitudinal oscillations the dispersion relation

$$\omega^2 = \omega_p{}^2 + 3\frac{\Theta}{m}k^2, \tag{30}$$

while for transverse oscillations we obtain

$$(\omega^2 - c^2k^2)\left(\omega^2 - \frac{\Theta}{m}k^2\right) + \omega^2\omega_p{}^2 = 0. \tag{31}$$

Eq. (31) leads to a quadratic in ω^2 whose roots are given by

$$\omega^2 = \frac{1}{2}\left(\omega_p^{\,2} + c^2 k^2 + \frac{k^2 \Theta}{m}\right) \pm \frac{1}{2}\left(\omega_p^{\,2} + c^2 k^2 + \frac{k^2 \Theta}{m}\right) \left\{1 - \frac{4\, c^2 k^2 k^2 \Theta/m}{[\omega_p^{\,2} + c^2 k^2 + (k^2 \Theta/m)]^2}\right\}^{1/2}. \tag{32}$$

In our approximation $\omega^2 \gg k^2 \Theta/m$. Retaining terms to the lowest order in $k^2 \Theta/m$, we obtain for the two transverse modes of oscillation (distinguished by the subscripts 1 and 2):

$$\omega_1^{\,2} = \omega_p^{\,2} + c^2 k^2 + \frac{k^2 \Theta}{m} \frac{\omega_p^{\,2}}{\omega_p^{\,2} + c^2 k^2} \tag{33}$$

and

$$\omega_2^{\,2} = \frac{k^2 \Theta}{m} \frac{1}{1 + (\omega_p^{\,2}/c^2 k^2)} \tag{34}$$

The root $\omega_1^{\,2}$ corresponds to the transverse oscillation; however, the root $\omega_2^{\,2}$ is incompatible with our basic approximation that $\omega^2 > k^2 \Theta/m$ and therefore must be discarded.

It may be remarked here that if, instead of using the equation for the stress tensor, we had used the classical adiabatic relation

$$\frac{d}{dt}(p\,\varrho^{-\gamma}) = 0, \tag{35}$$

we would have obtained for longitudinal oscillations the dispersion relation

$$\omega^2 = \omega_p^{\,2} + \gamma \frac{\Theta}{m} k^2 \tag{36}$$

instead of the relation (30). However, in the absence of collisions, it is necessary to allow for anisotropy in the pressure during the oscillations.

The effect of thermal motions on electron plasma oscillations was first considered by Thomson and later, by Bailey. They obtained the dispersion formula

$$\omega^2 = \omega_p^{\,2} + \frac{\Theta}{m} k^2 \tag{37}$$

for longitudinal oscillations. It can be seen readily that the appearance of the factor $\Theta k^2/m$ instead of $3\,\Theta k^2/m$ or $\gamma\,\Theta k^2/m$ results from neglecting the change in temperature of the gas resulting from the density fluctuations. This amounts to considering the oscillations to be isothermal and not adiabatic. This is incorrect for the high frequency electron plasma oscillations; for a consistent approximation it is necessary to use the complete set of equations (20)—(22) to treat the problem.

In part II of this survey this problem will be discussed from the standpoint of kinetic theory, and it will be seen that expressions (30) and (33) determine correctly to first order in Θ the real part of ω. The higher order thermal corrections can be recovered by going to higher order moment equations. However, an essential consequence of the treatment from the kinetic equation in the absence of collisions, which is not recovered on using the truncated set of moment equations devoid of collision terms, is the phenomena of Landau damping. That is, the kinetic equation yields for isotropic equilibria a negative imaginary part effectively proportional to the number of particles moving with the phase velocity of the wave. This number, of course, tends to be exponentially small since, by assumption, the thermal spread is much less than the phase velocity in question.

It must be emphasized here that the longitudinal and transverse oscillations are strictly uncoupled only in the case of a non-relativistic plasma and in the absence of any external magnetic fields, temperature or density gradients. The presence of an external magnetic field or inhomogeneities in plasma density and/or temperature result in the coupling of longitudinal and transverse modes, and the behaviour of the plasma, in general, is quite complex. This will be discussed in detail later.

2.4 Ion Oscillations

So far we have considered only electron plasma oscillations on the assumption that these oscillations are too rapid for the heavy ions to follow, which implies that the ions, therefore, may be considered at rest. Another class of oscillations, which is possible in a collision-free plasma, is the so-called ion oscillations. These are so slow that electrons see them as quasi static and consequently are distributed according to the Boltzmann distribution. During the oscillations it is appropriate to treat the ions and electrons as having different temperatures. However, we shall restrict ourselves to the case of longitudinal oscillations only.

The linearized equations of motion now are (cf. Eqs. (II-44)—(II-46)):

$$\frac{\partial n_i}{\partial t} + N \nabla \cdot \mathbf{v} = 0, \tag{38}$$

$$M N \frac{\partial \mathbf{v}}{\partial t} = -\nabla \cdot \mathsf{p}_i + N e \mathbf{E}, \tag{39}$$

$$\frac{\partial}{\partial t} \mathsf{p}_i = -p_i \nabla \cdot \mathbf{v}\, \mathbf{I} - p_i [\nabla \mathbf{v} + (\nabla \mathbf{v})^T], \tag{40}$$

$$\nabla \cdot \mathbf{E} = 4 \pi e (n_i - n_e), \tag{41}$$

and

$$n_e = N\left(\exp \frac{e \varphi}{\Theta_e} - 1\right) \tag{42}$$

where n_i and n_e denote the perturbation in the ion and electron equilibrium densities (assumed equal) respectively, Θ_e is the electron temperature; $p_i = N\Theta_i$ where Θ_i is the ion temperature, p_i is the perturbation in the ion material stress tensor, and M is the ion mass. φ denotes the electrostatic potential defined as $\mathbf{E} = -\nabla\varphi$. Following the normal mode procedure outlined in § 2.3 of this section, we obtain for the dispersion relation

$$\omega^2 = \frac{1}{1 + (k\lambda_D)^{-2}} \omega_{pi}^{\,2} + 3 k^2 \frac{\Theta_i}{M}, \tag{43}$$

where ω_{pi}, the ion plasma frequency, is given by

$$\omega_{pi} = \left(\frac{4 \pi N e^2}{M}\right)^{1/2} \tag{44}$$

and we recall that λ_D, the Debye shielding distance, is given by

$$\lambda_D = \left(\frac{\Theta_e}{4 \pi N e^2}\right)^{1/2} \tag{45}$$

Physically the reciprocal of λ_D acts like a screening radius for electrical forces in ionized media. Thus the potential in the neighborhood of a sphere charged with q units of electricity is effectively given by

$$\varphi = \frac{q}{r} \exp\left(-\lambda_D r\right) \tag{46}$$

in an ionized medium near thermal equilibrium, compared to q/r in a dielectric.

If $k\lambda_D \ll 1$, i.e., the wavelength of the disturbance is large compared to the Debye length, formula (43) reduces to

$$\omega^2 = \omega_{pi}{}^2 k^2 \lambda_D{}^2 + 3 k^2 \frac{\Theta_i}{M} = \frac{1}{M} (\Theta_e + 3\,\Theta_i)\, k^2 . \tag{47}$$

It is clear that if the ion temperature is very small compared to the electron temperature, the effect of the ion thermal motions can be neglected. We shall show in part II, using the kinetic description, that if Θ_i is not small compared with Θ_e, the ion oscillations are strongly damped. It may also be observed from Eq. (44) that the ion plasma frequency is much smaller than the electron plasma frequency.

The low frequency ion oscillations differ from the sound waves in the mechanism responsible for the organized motion. Ion oscillations are produced by long range Coulomb forces whereas sound waves are produced by short range collisions.

3. Unbounded Cold Plasmas in the Presence of an External Magnetic Field

We shall now consider oscillations in a uniform plasma in the presence of a constant external magnetic field B_0. We shall assume the plasma to be at zero temperature; this amounts to neglecting the thermal motions of the ions and electrons. The effect of thermal motions will be taken into account quite rigorously via the Boltzmann equation in part II. We shall further assume that each constituent of the plasma has in the equilibrium state a constant velocity $\mathbf{u}$. The motion of the constituents of the plasma will, in general, give rise to a current distribution $\mathbf{J}$. We shall, however, assume that the magnetic field which results from $\mathbf{J}$ is negligibly small compared with the external magnetic field.

The linearized equations of motion for each of the species of the plasma are*:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{u} \cdot \nabla)\, \mathbf{v} = \frac{e}{m} \left(\mathbf{E} + \frac{1}{c}\, \mathbf{u} \times \mathbf{B} + \frac{1}{c}\, \mathbf{v} \times \mathbf{B}_0 \right), \tag{48}$$

$$\frac{\partial n}{\partial t} + \nabla \cdot (N \mathbf{v} + n\, \mathbf{u}) = 0, \tag{49}$$

where the perturbations in the electric ($\mathbf{E}$) and magnetic ($\mathbf{B}$) fields are given by the Maxwell equations

$$\nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{J} + \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t}, \tag{50}$$

$$\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}, \tag{51}$$

and

$$\mathbf{J} = \sum e\, (N \mathbf{v} + n\, \mathbf{u}) . \tag{52}$$

The summation Eq. (52) is to be carried over both the species of the plasma. The cyclotron frequency is defined to be

$$\omega_c = \frac{e B_0}{m c} . \tag{53}$$

We shall now assume that all quantities have space and time dependence of the form $\exp i\,(-\omega t + \mathbf{k} \cdot \mathbf{r})$. Eqs. (48) and (49) then yield

$$n = \frac{N}{(\omega - \mathbf{k} \cdot \mathbf{u})} (\mathbf{k} \cdot \mathbf{v}) \tag{54}$$

and

$$-i\,(\omega - \mathbf{k} \cdot \mathbf{u})\, \mathbf{v} = \frac{e}{m} \left(\mathbf{E} + \frac{1}{c}\, \mathbf{u} \times \mathbf{B} \right) - \boldsymbol{\omega}_c \times \mathbf{v} . \tag{55}$$

The appearance of the factor $\omega - \mathbf{k} \cdot \mathbf{u}$ instead of ω is typical of a current-carrying plasma; the term $\mathbf{k} \cdot \mathbf{u}$ accounts just for the Doppler shift in the frequency ω due to the motion of the electrons or the ions. From Eqs. (51) and (55) we obtain

$$-i\,(\omega - \mathbf{k} \cdot \mathbf{u})\, (\boldsymbol{\omega}_c \cdot \mathbf{v}) = \frac{e}{m} (\boldsymbol{\omega}_c \cdot \mathbf{E}) + \frac{e}{m\,\omega}\, \boldsymbol{\omega}_c \cdot [\mathbf{u} \times (\mathbf{k} \times \mathbf{E})] \tag{56}$$

and

$$-i\,(\omega - \mathbf{k} \cdot \mathbf{u})\, (\boldsymbol{\omega}_c \times \mathbf{v}) = \frac{e}{m} (\boldsymbol{\omega}_c \times \mathbf{E}) + \frac{e}{m\,\omega}\, \boldsymbol{\omega}_c \times [\mathbf{u} \times (\mathbf{k} \times \mathbf{E})] - \boldsymbol{\omega}_c\, (\boldsymbol{\omega}_c \cdot \mathbf{v}) + \omega_c{}^2\, \mathbf{v} . \tag{57}$$

Using the results (56) and (57) into Eq. (55), we obtain

$$\begin{aligned} \mathbf{v} = -\frac{e}{m} \frac{1}{\omega_c{}^2 - (\omega - \mathbf{k} \cdot \mathbf{u})^2} & \left[i\,(\omega - \mathbf{k} \cdot \mathbf{u}) \left\{ \mathbf{E} + \frac{1}{\omega}\, \mathbf{u} \times (\mathbf{k} \times \mathbf{E}) \right\} \right. \\ & + \boldsymbol{\omega}_c \times \mathbf{E} + \frac{1}{\omega}\, \boldsymbol{\omega}_c \times \left\{ \mathbf{u} \times (\mathbf{k} \times \mathbf{E}) \right\} \\ & \left. - i\, \frac{\boldsymbol{\omega}_c}{(\omega - \mathbf{k} \cdot \mathbf{u})} \left\{ (\boldsymbol{\omega}_c \cdot \mathbf{E}) + \frac{1}{\omega}\, \boldsymbol{\omega}_c \cdot [\mathbf{u} \times (\mathbf{k} \times \mathbf{E})] \right\} \right] . \end{aligned} \tag{58}$$

From Eqs. (52) and (54) we obtain for the current density due to oscillations

$$\mathbf{J} = \sum e N \left(\mathbf{v} + \mathbf{u}\, \frac{\mathbf{k} \cdot \mathbf{v}}{\omega - \mathbf{k} \cdot \mathbf{u}} \right) . \tag{59}$$

Equations (50), (51), and (59) lead to

$$(\omega^2 - c^2 k^2)\, \mathbf{E} + c^2\, \mathbf{k}\, (\mathbf{k} \cdot \mathbf{E}) + 4\pi i\, \omega \sum e N \left(\mathbf{v} + \mathbf{u}\, \frac{\mathbf{k} \cdot \mathbf{v}}{\omega - \mathbf{k} \cdot \mathbf{u}} \right) = 0 , \tag{60}$$

where $\mathbf{v}$ is given by Eq. (58). On substituting for $\mathbf{v}$ in accordance with Eq. (58) into Eq. (60) we obtain, in general, an equation of the form

$$\mathbf{R} \cdot \mathbf{E} = 0 , \tag{61}$$

---

* The notation used here ($\mathbf{u}$ for the velocity in the steady state and $\mathbf{v}$ for the perturbation in the velocity) is different from that used in section II. This should, however, cause no confusion.

where **R** is a 3×3 Hermitian matrix. In order that Eq. (61) have a non-trivial solution, we must demand that the secular equation

$$\| \mathbf{R} \| = 0\,, \tag{62}$$

be satisfied. This results in the required dispersion relation. We shall now discuss the implications of Eq. (62) in several special cases.

3.1 Oscillations in a Static Plasma

We shall first discuss the case of oscillations in a static plasma, i.e., where the mean fluid velocity vanishes everywhere. Setting $\mathbf{u} = 0$ in Eq. (58) and substituting the resulting expression into Eq. (60), we obtain an equation of the form (61). Taking $\mathbf{k}$ to be along the z-axis of a Cartesian system of co-ordinates and assuming that the magnetic field lies in the xz plane making an angle θ with the z-axis, the elements of the matrix **R** are given by

$$\left.\begin{aligned}
R_{11} &= \omega^2 - c^2 k^2 - \sum \omega_p{}^2 + \sum \frac{\omega_p{}^2 \omega_c{}^2 \cos^2\theta}{\omega_c{}^2 - \omega^2} \\
R_{12} &= - R_{21} = \sum i\, \omega_p{}^2 \frac{\omega\, \omega_c \cos\theta}{\omega_c{}^2 - \omega^2} \\
R_{13} &= + R_{31} = - \sum \omega_p{}^2 \frac{\omega_c{}^2}{\omega_c{}^2 - \omega^2} \sin\theta\cos\theta \\
R_{22} &= \omega^2 - c^2 k^2 - \sum \omega_p{}^2 + \sum \omega_p{}^2 \frac{\omega_c{}^2}{\omega_c{}^2 - \omega^2} \\
R_{23} &= - R_{32} = i \sum \omega_p{}^2 \frac{\omega\omega_c \sin\theta}{\omega_c{}^2 - \omega^2} \\
R_{33} &= \omega^2 - \sum \omega_p{}^2 + \sum \omega_p{}^2 \frac{\omega_c{}^2 \sin^2\theta}{\omega_c{}^2 - \omega^2}
\end{aligned}\right\}, \tag{63}$$

where $\omega_p{}^2 = 4\,\pi\, Ne^2/m$.

3.11 No Magnetic Field Case

As the simplest illustrative example of Eq. (62), consider the case of electron oscillations. Setting $\omega_c = 0$ in Eq. (63), we obtain Eq. (64), see below on this page, where ω_p now denotes the electron plasma frequency. We thus obtain for longitudinal oscillations (**k** along **E**)

$$\omega^2 = \omega_p{}^2\,, \tag{65}$$

and for transverse oscillations (**k** perpendicular to **E**)

$$\omega^2 = \omega_p{}^2 + c^2 k^2\,, \tag{66}$$

which are in accordance with Eqs. (8) and (17).

3.12 Propagation along the Magnetic Field

For propagation along the magnetic field, $\theta = 0$, and Eq. (61) takes the particularly simple form (cf. Eqs. (61) and (63)) of Eq. (67), see bottom of this page.

For longitudinal oscillations we thus obtain the relation

$$\omega^2 = \sum \omega_p{}^2 = \omega_{pe}{}^2 \left(1 + \frac{m}{M}\right) \simeq \omega_{pe}{}^2\,, \tag{68}$$

where ω_{pe} denotes the electron plasma frequency. The transverse oscillations are circularly polarized and are described by the equation

$$\left[\omega^2 - c^2 k^2 - \sum \omega_p{}^2 \frac{\omega}{\omega \mp \omega}\right] E_{\pm} = 0, \tag{69}$$

where

$$E_{\pm} = E_x \pm i E_x\,, \tag{70}$$

the plus or minus sign corresponding to the right or left handed circularly polarized waves respectively. The oscillations are, therefore, governed by the dispersion relation

$$\begin{aligned}
\omega^2 - c^2 k^2 - \omega_{pe}{}^2 - \omega_{pi}{}^2 + \omega_{pe}{}^2 \frac{\omega_e}{\omega + \omega_e} \\
- \omega_{pi}{}^2 \frac{\omega_i}{\omega - \omega_i} = 0.
\end{aligned} \tag{71}$$

for the right handed circularly polarized waves. Here the electron and the ion plasma frequencies are denoted by ω_{pe} and ω_{pi} respectively, while the electron and ion cyclotron frequencies are denoted by ω_e and ω_i respectively. A relation similar to (71) applies to the left handed circularly polarized waves.

a. High Frequency Oscillations

If $\omega \gg \omega_i$, the dispersion relation (71) reduces to

$$\omega^2 - c^2 k^2 - \omega_{pe}{}^2 \frac{\omega}{\omega + \omega_e} - \omega_{pi}{}^2 = 0\,. \tag{72}$$

In the limit when $\omega \gg \omega_e$, Eq. (72) yields, correct to the lowest significant order in ω_e/ω,

$$\omega^2 = c^2 k^2 + \omega_{pe}{}^2 \left[1 - \frac{\omega_e}{(\omega_{pe}{}^2 + c^2 k^2)^{1/2}}\right]. \tag{73}$$

It is clear that a necessary condition for the validity of this relation is that $\omega_e{}^2 \ll \omega_{pe}{}^2 + c^2k^2$.

$$\begin{vmatrix}
\omega^2 - \omega_p{}^2 - c^2 k^2 & 0 & 0 \\
0 & \omega^2 - \omega_p{}^2 - c^2 k^2 & 0 \\
0 & 0 & \omega^2 - \omega_p{}^2
\end{vmatrix}
\begin{vmatrix} E_x \\ E_y \\ E_z \end{vmatrix} = 0 \tag{64}$$

$$\begin{vmatrix}
\omega^2 - c^2 k^2 + \sum \dfrac{\omega_p{}^2 \omega^2}{\omega_c{}^2 - \omega^2} & i \sum \omega_p{}^2 \dfrac{\omega\, \omega_c}{\omega_c{}^2 - \omega^2} & 0 \\
-i \sum \omega_p{}^2 \dfrac{\omega\, \omega_c}{\omega_c{}^2 - \omega^2} & \omega^2 - c^2 k^2 + \sum \omega_p{}^2 \dfrac{\omega^2}{\omega_c{}^2 - \omega^2} & 0 \\
0 & 0 & \omega^2 - \sum \omega_p{}^2
\end{vmatrix}
\begin{Vmatrix} E_x \\ E_y \\ E_z \end{Vmatrix} = 0. \tag{67}$$

b. Oscillations near the Electron Cyclotron Frequency

Another case of interest is the one when the frequency of oscillations is close to the electron cyclotron frequency. In this case we find that Eq. (72) leads to

$$\omega = -\omega_e \left[1 - \frac{\omega_{pe}^2}{\omega_e^2 - (c^2 k^2 + \omega_{pe}^2)}\right]. \qquad (74)$$

It is clear that the necessary condition for this approximation to be valid is that

$$\omega_{pe}^2 < | \omega_e^2 - c^2 k^2 |. \qquad (75)$$

We can write Eq. (72) in the form (for $\omega \gg \omega_i$)

$$\left(\frac{V}{c}\right)^2 = \frac{1 + (\omega_e/\omega)}{1 + (\omega_e/\omega) - (\omega_{pe}^2/\omega^2)}, \qquad (76)$$

where $V = \omega/k$ is the phase velocity of the wave. The behaviour of the phase velocity as a function of the frequency of oscillation is illustrated in Figs. 1—3 for some values of the field strength.

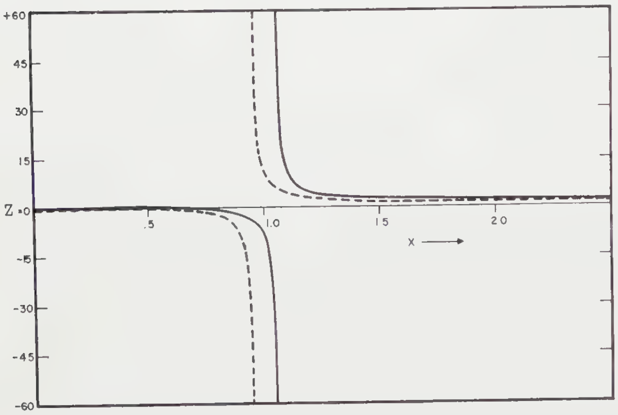

Fig. 1 The behavior of $Z = (V/c)^2$, where V is the phase velocity of the wave, as a function of $x = \omega/\omega_p$ for electron oscillations for propagation along the magnetic field for the case when $\omega_e = 0.1\,\omega_p$. The negative values of Z correspond to imaginary values of k, which correspond to absorption of waves in a bounded plasma.

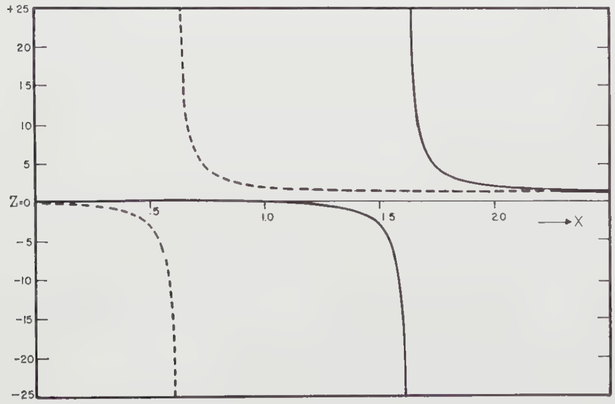

Fig. 2 The behavior of $Z = (V/c)^2$ as a function of $x = \omega/\omega_p$ for electron oscillations for propagation along the magnetic field for the case when $\omega_e = \omega_p$.

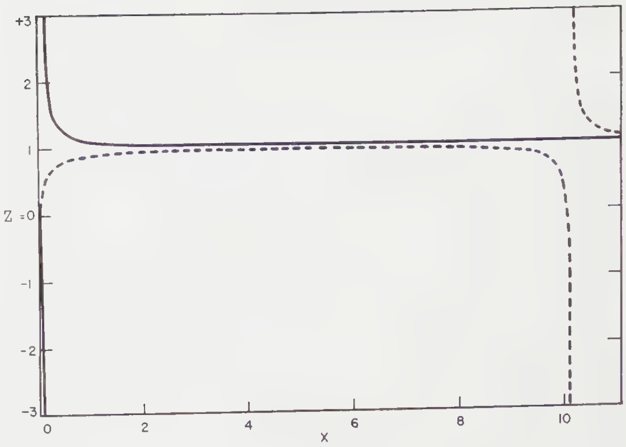

Fig. 3 The behavior of $Z = (V/c)^2$ as a function of $x = \omega/\omega_p$ for electron oscillations for propagation along the magnetic field for the case when $\omega = 10\,\omega_p$.

c. Oscillations much below the Ion Cyclotron Frequency

Now consider the low frequency oscillations such that the condition $\omega < \omega_i$ is satisfied. Then to the lowest significant order we can write Eq. (71) as

$$\omega^2 - c^2 k^2 + \omega_{pi}^2 \frac{\omega}{\omega_i}\left(1 + \frac{\omega}{\omega_i}\right) - \omega_{pe}^2 \frac{\omega}{\omega_e} = 0. \qquad (77)$$

Using the fact that $\omega_{pi}^2/\omega_i = \omega_{pe}^2/\omega_e$ and $\omega_{pi}/\omega_i^2 = c^2\, 4\pi NM/B_0^2$, Eq. (77) leads to

$$\omega^2 = \frac{c^2 k^2}{1 + (c^2/A^2)}, \qquad (78)$$

where

$$A = \frac{B_0}{(4\pi NM)^{1/2}} \qquad (79)$$

denotes the Alfvén speed. Eq. (78) is the dispersion relation of Aström for the extra-ordinary hydromagnetic wave. It must be emphasized here that the hydromagnetic wave, in principle, is just a special case, in the appropriate frequency region, of the well-known transverse electromagnetic waves.

In the limit when $c/A \gg 1$, Eq. (78) reduces to

$$\left(\frac{\omega}{k}\right)^2 = A^2; \qquad (80)$$

that is, the phase velocity of the wave is just the Alfvén speed. In the other limit when $c/A \ll 1$, we get

$$\omega^2 = c^2 k^2, \qquad (80\,a)$$

which corresponds to the usual electromagnetic modes.

d. Oscillations near the Ion Cyclotron Frequency

Let us now consider the case when $\omega \simeq \omega_i$. This implies that $\omega < \omega_e$, and the dispersion relation (71) reduces to

$$\omega^2 - c^2 k^2 - \omega_{pi}^2 - \omega_{pi}^2 \frac{\omega_i}{\omega - \omega_i} = 0. \qquad (81)$$

This leads to the frequency of oscillation

$$\omega \simeq \omega_i \left[1 + \frac{\omega_{pi}^2}{\omega_i^2 - (c^2 k^2 + \omega_{pi}^2)}\right]. \tag{81 a}$$

The condition for the validity of this approximation is that

$$\omega_i^2 \gg \omega_{pi}^2 + c^2 k^2 .$$

3.13 Propagation Transverse to the Magnetic Field

For propagation perpendicular to the magnetic field, $\theta = \pi/2$, and we obtain the secular Eq. (82) see below. One of the modes of oscillation corresponds to the usual transverse oscillation with electric field along $\mathbf{B}_0$ and satisfies the dispersion relation

$$\omega^2 = c^2 k^2 + \sum \omega_p^2 \simeq c^2 k^2 + \omega_{pe}^2 . \tag{83}$$

The other two modes of oscillation are determined as roots of the equation (84), see bottom of this page.

Consider the case when $\omega \ll ck$. Then to the lowest significant order, the 22 element of this determinant must vanish. We thus obtain

$$1 + \frac{\omega_{pi}^2}{\omega_i^2 - \omega^2} + \frac{\omega_{pe}^2}{\omega_e^2 - \omega^2} = 0 . \tag{85}$$

In the limit when $\omega_i \ll \omega \ll \omega_e \ll \omega_p$, Eq. (85) reduces to

$$-\frac{\omega_{pi}^2}{\omega^2} + \frac{\omega_{pe}^2}{\omega_e^2} = 0. \tag{86}$$

The frequency of oscillation is then determined by

$$\omega = \left(\frac{\omega_e^2\, \omega_{pi}^2}{\omega_{pe}^2}\right)^{1/2} = (\omega_e\, \omega_i)^{1/2} \tag{87}$$

This frequency of oscillation is referred to as the *lower hybrid frequency.*

Now consider the limiting case where the conditions $\omega^2 \gg \omega_i^2$ and $\omega^2 \gg \omega_{pi}^2$ are satisfied. Eq. (85) then leads to

$$\omega = (\omega_e^2 + \omega_{pe}^2)^{1/2}. \tag{88}$$

This frequency of oscillation is referred to as the *upper hybrid frequency.*

For the high frequency oscillations, $\omega \gg \omega_i, \omega_{pi}$, we may neglect the motion of the ions, and Eq. (84) leads to

$$\left(\omega^2 + \omega_{pe}^2 \frac{\omega^2}{\omega_e^2 - \omega^2}\right)\left(\omega^2 + \omega_{pe}^2 \frac{\omega^2}{\omega_e^2 - \omega^2} - c^2 k^2\right) - \omega_{pe}^4 \frac{\omega^2 \omega_e^2}{\omega_e^2 - \omega^2} = 0. \tag{89}$$

The behaviour of the phase velocity as a function of the frequency corresponding to the roots of this equation are illustrated in Figs. 4—6 for some values of the field strength.

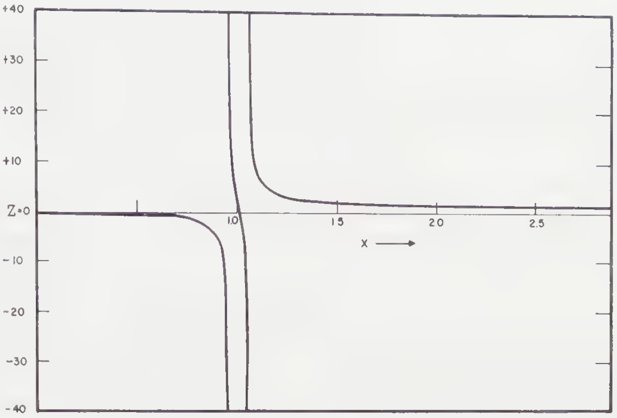

Fig. 4 The behavior of $Z = (V/c)^2$, where V denotes the phase velocity of the wave, as a function of $x = \omega/\omega_p$ for electron oscillations for propagation transverse to the magnetic field for the case when $\omega_e = 0.1\ \omega_p$.

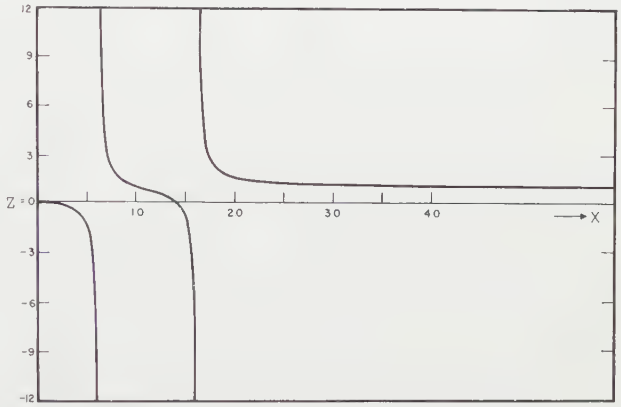

Fig. 5 The behavior of $Z = (V/c)^2$, where V denotes the phase velocity of the wave as a function of $x = \omega/\omega_p$ for electron oscillations for propagation transverse to the magnetic field for the case when $\omega_e = \omega_p$

3.14 Weak Magnetic Field

We now consider the effect of a weak magnetic field on the high frequency electron oscillations described

$$\begin{vmatrix} \omega^2 - \sum \omega_p^2 - c^2 k^2 & 0 & 0 \\ 0 & \omega^2 - c^2 k^2 + \sum \frac{\omega_p^2}{\omega_c^2 - \omega^2} & -i \sum \omega_p^2 \frac{\omega\omega_c}{\omega_c^2 - \omega^2} \\ 0 & +i \sum \omega_p^2 \frac{\omega\omega_c}{\omega_c^2 - \omega^2} & \omega^2 + \sum \omega_p^2 \frac{\omega^2}{\omega_c^2 - \omega^2} \end{vmatrix} = 0. \tag{82}$$

$$\begin{vmatrix} \omega^2 - c^2 k^2 + \sum \omega_p^2 \frac{\omega^2}{\omega_c^2 - \omega^2} & -i \sum \omega_p^2 \frac{\omega\omega_c}{\omega_c^2 - \omega^2} \\ +i \sum \omega_p^2 \frac{\omega\omega_c}{\omega_c^2 - \omega^2} & \omega^2 + \sum \omega_p^2 \frac{\omega^2}{\omega_c^2 - \omega^2} \end{vmatrix} = 0. \tag{84}$$

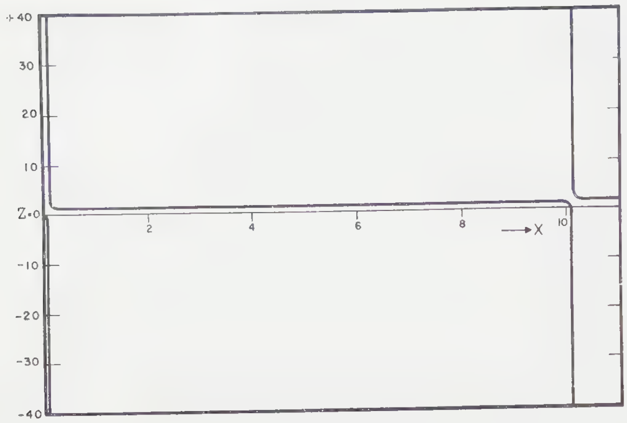

Fig. 6 The behavior of $Z = (V/c)^2$, where V denotes the phase velocity of the wave as a function of $x = \omega/\omega_p$ for electron oscillations for propagation transverse to the magnetic field for the case when $\omega_e = 10\,\omega_p$.

in § 2 for arbitrary direction of the magnetic field. The weak field approximation, specifically, implies that $\omega_e \ll \omega$. Retaining terms up to second order in ω_c/ω, the dispersion equation now becomes (cf. Eqs. (61) and (63)) Eq. (90) see bottom of this page, where we have put

$$\omega_0^2 = \omega_p^2 + c^2 k^2; \tag{91}$$

ω_0 is the frequency of oscillation of the transverse wave in the absence of a magnetic field. Since we have assumed ω_e/ω to be small, to a first approximation, for the transverse oscillations, ω can be replaced by ω_0 in all elements of the matrix **R** except R_{11} and R_{22}. Then it is clear that R_{33} is the leading term in **R**, being of order c^2k^2 and Eq. (90) gives

$$c^2 k^2 E_z = 0\,. \tag{92}$$

Since $k \neq 0$, we must require that E_z vanish. Thus to first approximation in ω_e/ω, the longitudinal and transverse modes of oscillation remain uncoupled. Retaining terms up to first order in ω_e/ω, Eq. (90) reduces to

$$\begin{vmatrix} \omega^2 - \omega_0^2 & -i\omega_{pe}^2 \frac{\omega_e}{\omega_0}\cos\theta \\ i\omega_{pe}^2 \frac{\omega_e}{\omega_0}\cos\theta & \omega^2 - \omega_0^2 \end{vmatrix} \begin{vmatrix} E_x \\ E_y \end{vmatrix} = 0, \tag{93}$$

In order that this equation have a non-trivial solution, we must demand that the determinant of the co-efficients vanish; this leads to the dispersion relation

$$\omega^2 = \omega_0^2 \pm \omega_{pe}^2 \frac{\omega_e}{\omega_0}\cos\theta\,. \tag{94}$$

These modes of oscillation are circularly polarized with $E_x = \pm\, iE_y$. It can be verified readily from Eqs. (90) and (94) that to first order in ω_e/ω, E_z vanishes.

To carry through the calculations consistent to the second order in ω_e/ω, we now substitute ω in accordance with Eq. (94) in all elements of the matrix **R** except R_{11} and R_{22} in Eq. (90). Retaining terms up to second order in ω_e/ω, we obtain

$$\begin{aligned} \omega^2 = \omega_0^2 + \frac{1}{2}\omega_{pe}^2 \frac{\omega_e^2}{\omega_0^2}(1+\cos^2\theta) \pm \\ \frac{\omega_{pe}^2\omega_e}{2\omega_0}\Bigg[4\cos^2\theta\left\{1 \mp \frac{\omega_{pe}^2}{\omega_0^2}\frac{\omega_e}{\omega_0}\cos\theta + \frac{\omega_{pe}^4}{\omega_0^4}\frac{\omega_e^2}{\omega_0^2}\cos^2\theta\right\} \\ + \left(\frac{\omega_e}{\omega_0}\right)^2 \Bigg\{(1-\cos\theta)^2 + \sin^2\theta\,\frac{\omega_{pe}^2}{c^2k^2}\Bigg[2(1+\cos^2\theta) \\ + \sin^2\theta\,\frac{\omega_{pe}^2}{c^2k^2}\Bigg]\Bigg\}\Bigg]^{1/2}. \end{aligned} \tag{95}$$

In particular, for $\theta = 0$, we obtain

$$\omega^2 = \omega_0^2 \pm \omega_{pe}^2 \frac{\omega_e}{\omega_0} + \omega_{pe}^2 \frac{\omega_e^2}{\omega_0^2}\left(1 \pm \frac{1}{2}\frac{\omega_{pe}^2}{\omega_0^2}\right), \tag{96}$$

and for $\theta = \pi/2$, Eq. (95) gives

$$\omega^2 = \omega_0^2 - \frac{1}{2}\omega_{pe}^2 \frac{\omega_e^2}{\omega_0^2}\left[1 \pm \left(1 + \frac{\omega_{pe}^2}{c^2k^2}\right)\right]. \tag{97}$$

We now find that the longitudinal and transverse waves are coupled; their polarizations are given by

$$\frac{E_x}{E_y} = \mp i\left(1 \pm \frac{1}{2}\frac{\omega_{pe}^2}{\omega_0^2}\frac{\omega_e}{\omega_0}\cos\theta\right) \tag{98}$$

and

$$\frac{E_z}{E_y} = -i\,\frac{1}{c^2k^2}\,\omega_{pe}^2\,\frac{\omega_e}{\omega_0}\sin\theta \tag{99}$$

Let us now consider longitudinal oscillations which are characterized by the plasma frequency $\omega = \omega_{pe}$ in the absence of a magnetic field. To a first approximation we can put $\omega = \omega_{pe}$ in all elements of the matrix **R** except R_{33} and obtain the secular equation (100) below.

$$\begin{vmatrix} \omega^2 - \omega_0^2 - \omega_{pe}^2 \frac{\omega_e^2}{\omega^2}\cos^2\theta & -i\omega_{pe}^2\frac{\omega_e}{\omega}\cos\theta & \omega_{pe}^2\frac{\omega_e^2}{\omega^2}\sin\theta\cos\theta \\ i\omega_{pe}^2\frac{\omega_e}{\omega}\cos\theta & \omega^2 - \omega_0^2 - \omega_{pe}^2\frac{\omega_e^2}{\omega^2} & -i\omega_{pe}^2\frac{\omega_e}{\omega}\sin\theta \\ \omega_{pe}^2\frac{\omega_e^2}{\omega^2}\sin\theta\cos\theta & +i\omega_{pe}^2\frac{\omega_e}{\omega}\sin\theta & \omega^2 - \omega_{pe}^2 - \omega_{pe}^2\frac{\omega_e^2}{\omega^2}\sin^2\theta \end{vmatrix} \begin{vmatrix} E_x \\ E_y \\ E_z \end{vmatrix} = 0, \tag{90}$$

$$\begin{vmatrix} -c^2k^2 - \omega_e^2\cos^2\theta & -i\omega_{pe}\omega_e\cos\theta & \omega_e^2\sin\theta\cos\theta \\ i\omega_{pe}\omega_e\cos\theta & -c^2k^2 - \omega_e^2 & -i\omega_{pe}\omega_e\cos\theta \\ \omega_e^2\sin\theta\cos\theta & i\omega_{pe}\omega_e\sin\theta & \omega^2 - \omega_{pe}^2 - \omega_e^2\sin^2\theta \end{vmatrix} = 0\,. \tag{100}$$

The electrostatic oscillations are characterized by the condition that $ck \gg \omega$. Thus the leading elements of the matrix are R_{11} and R_{22}, and to compute the corrections to the root thus obtained, it can be readily verified that it is sufficient to demand that the co-factor of R_{11} vanish. This yields the dispersion relation

$$\omega^2 = \omega_{pe}^2 \left[1 + \left(\frac{\omega_e^2}{\omega_{pe}^2} + \frac{\omega_e^2}{c^2k^2}\right)\sin^2\theta\right] \qquad (101)$$

The correction terms are small as long as ω_e^2/ω_{pe}^2 and ω_e^2/c^2k^2 are both much less than unity.

It may be remarked here that for transverse oscillations the correction terms to ω^2, to lowest significant order, are of order ω_e/ω, while for longitudinal oscillations the correction terms, to the lowest significant order, are of order $(\omega_e/\omega)^2$.

3.15 Strong Magnetic Field

We shall now consider the case of a strong magnetic field which implies that $\omega_i/\omega \gg 1$. Retaining terms up to second order in ω_c/ω, Eqs. (61) and (63) lead to (102), see bottom of page. In writing Eq. (102) we have used

$$\sum \frac{\omega_p^2}{\omega_c^2} = \frac{4\pi NMc^2}{B_0^2}\left(1 + \frac{m}{M}\right) \simeq \frac{4\pi NMc^2}{B_0^2} = \frac{c^2}{A^2}, \qquad (103)$$

and

$$\frac{\omega_{pe}^2}{\omega_e} = \frac{\omega_{pi}^2}{\omega_i}. \qquad (104)$$

One of the roots of Eq. (102) is clearly given by

$$\omega^2 = \frac{c^2k^2}{1 + (c^2/A^2)}, \qquad (105)$$

this is the dispersion formula of Aström for the extraordinary hydromagnetic wave. The characteristic frequencies of the other two modes are determined by the algebraic equation (assuming $c/A \gg 1$)

$$\frac{c^4}{A^4}\omega^4 - \frac{c^2}{A^2}\omega^2(c^2k^2 + \omega_{pe}^2) + c^2k^2\omega_{pe}^2\cos^2\theta = 0. \qquad (106)$$

The roots of this equation are given by

$$\omega^2 = \frac{A^2}{2c^2}\Big\{c^2k^2 + \omega_{pe}^2 + [(c^2k^2 + \omega_{pe}^2)^2 - 4c^2k^2\omega_{pe}^2\cos^2\theta]^{1/2}. \qquad (107)$$

In hydromagnetic waves $\omega_{pe} \ll ck$, and Eq. (107) reduces to

$$\left(\frac{\omega}{k}\right)^2 = A^2; \qquad (108)$$

this is the ordinary hydromagnetic wave of Alfvén.

4. Instability in a Plasma Carryine Current

We now consider the case of a uniform plasma composed of an equal number of electrons and protons in the absence of any external magnetic field. The ions will be assumed to be at rest, and the electrons, moving with a constant velocity u. We shall neglect the magnetic field which results from the electron current, a procedure which is plausible if the associated magnetic forces are much less than the electric forces considered. For the problem at hand, Eqs. (58) and (60) reduce to

$$(\omega^2 - c^2k^2)\mathbf{E} + c^2\mathbf{k}(\mathbf{k}\cdot\mathbf{E}) + 4\pi i\omega \Sigma eN\left(\mathbf{v} + \mathbf{u}\frac{\mathbf{k}\cdot\mathbf{v}}{\omega - \mathbf{k}\cdot\mathbf{u}}\right) = 0. \qquad (109)$$

and

$$\mathbf{v} = \frac{ie}{m}\frac{1}{\omega - \mathbf{k}\cdot\mathbf{u}}\left[\mathbf{E} + \frac{1}{\omega}\mathbf{u}\times(\mathbf{k}\times\mathbf{E})\right]. \qquad (110)$$

We first consider electrostatic oscillations only and take $\mathbf{k}$ to be in the direction of the electron stream. Eqs. (109) and (110) then lead to the dispersion relation

$$1 = \frac{\omega_{pi}^2}{\omega^2} + \frac{\omega_{pe}^2}{(\omega - ku)^2}, \qquad (111)$$

where ω_{pi} and ω_{pe} denote the ion and electron plasma frequencies respectively. It is now convenient to introduce the dimensionless parameters

$$x = \frac{\omega}{\omega_{pe}}, \qquad y = \frac{ku}{\omega_{pe}}, \qquad \text{and} \quad \varepsilon = \frac{m}{M} \qquad (112)$$

The dispersion relation (111) then takes the form

$$F(x, y) = \frac{1}{(x - y)^2} + \frac{\varepsilon}{x^2} = 1. \qquad (113)$$

A schematic plot of $F(x, y)$ as a function of x for $y = 1$ is shown Fig. 7 for a hydrogen plasma. If the line $F = 1$ lies above the minimum of the curve, then it is

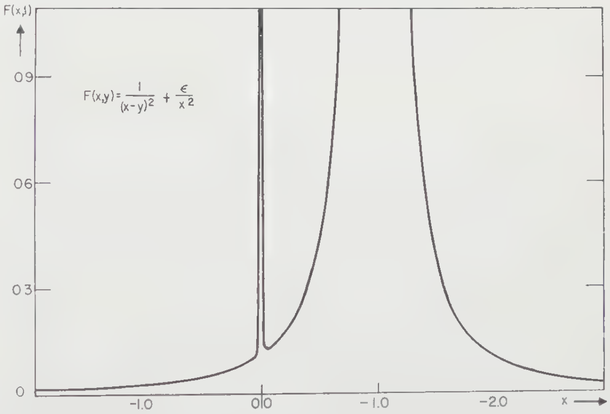

Fig. 7 The plot of $F(x, y)$ given by Eq. (113) as a function of x for $y = 1$ for a plasma composed of electrons and protons.

$$\begin{vmatrix} \omega^2 - c^2k^2 + \frac{c^2}{A^2} - \omega_{pe}^2\sin^2\theta & 0 & -\omega_{pe}^2\sin\theta\cos\theta \\ 0 & \omega^2 - c^2k^2 + \frac{c^2}{A^2}\omega^2 & 0 \\ -\omega_{pe}^2\sin\theta\cos\theta & 0 & \omega^2 + \frac{c^2}{A^2} - \omega_{pe}^2\cos^2\theta \end{vmatrix} = 0, \qquad (102)$$

clear that there exist four real roots of Eq. (113) and that all modes of oscillation are oscillatory. However, if the line $F = 1$ lies below the minimum of the curve, then there are only two real roots and there must exist a pair of complex conjugate roots, one of which has a positive imaginary part and is, therefore, unstable. The minimum of the curve $F(x, y)$ is determined by the condition that, for a given y, $F(x, y)$ be stationary, i.e.,

$$\frac{\partial}{\partial x} F(x, y) = 0. \tag{114}$$

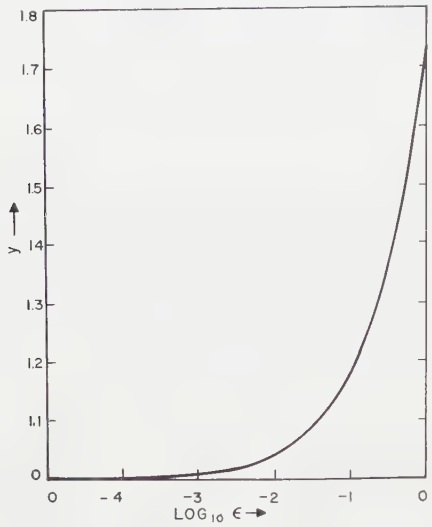

Fig. 8 The behavior of y_c given by Eq. (116) as a function of ε, the ratio of the negative to the positive ion mass.

This leads to

$$x = \frac{\varepsilon^{1/3}}{1 + \varepsilon^{1/3}} y \simeq \varepsilon^{1/3} y, \text{ if } \varepsilon \ll 1. \tag{115}$$

The value of y for which this minimum is just tangent to the line $F = 1$ can be readily obtained by substituting this value of x into Eq. (113) and solving the resulting equation for y. Denoting these minimum values by x_c and y_c, we have

$$y_c = [\varepsilon^{1/3} + (1 - \varepsilon^{1/3})^2]^{1/2}. \tag{116}$$

The behaviour of y_c as a function of ε is illustrated in Fig. 8. If $y_c > y$, Eq. (113) admits four real roots. Thus a cold streaming plasma is *stable* to all perturbations having wavelengths smaller than the critical wavelength λ_c given by

$$\lambda_c = 2\pi \frac{u}{\omega_{pe}} \left(1 + \frac{3}{2} \varepsilon^{1/3}\right)^{-1} \tag{117}$$

$$\sim 2\pi \frac{u}{\omega_{pe}} \left(1 - \frac{3}{2} \varepsilon^{1/3}\right), \quad \text{if } \varepsilon \ll 1. \tag{118}$$

For values of y close to y_c, by Taylor's expansion, Eq. (113) can be written as (cf. Eq. (114))

$$(y - y_c) F_y(x_c, y_c) + \frac{1}{2}(x - x_c)^2 F_{xx}(x_c, y_c) = 0. \tag{119}$$

This leads to

$$x = x_c \pm i \varepsilon^{1/6} \left[\frac{2}{3}(y_c - y)\right]^{1/2}; \tag{120}$$

it is clear that this expression is valid so long as

$$y_c - y \ll \frac{3}{2} \varepsilon^{1/3}. \tag{121}$$

Eq. (120) implies that

$$\operatorname{Im} \frac{dx}{dy} \to \pm \infty \quad \text{as } y \to y_c. \tag{122}$$

For values of y close to the origin, and on the assumption that $x/y \ll 1$, an approximation which is legitimate for the unstable roots (cf. Eq. (116)), we obtain

$$x = \pm i \varepsilon^{1/2} y. \tag{123}$$

Equations (122) and (123) give the behaviour of x near x_c and the origin as a function of y. For the continuous range of values of y between 0 and y_c, Eq. (113) has been solved numerically, and the results are illustrated in Fig. 9 where the real and imaginary parts of x are plotted as a function of y.

The growth rate of maximum instability can be readily computed from Eq. (113) by observing that at the maximum

$$\operatorname{Im} \frac{dx}{dy} = 0. \tag{124}$$

From Eq. (111) we obtain

$$\frac{dx}{dy} = \frac{1}{1 - \varepsilon \left(\frac{y}{x} - 1\right)^3} \tag{125}$$

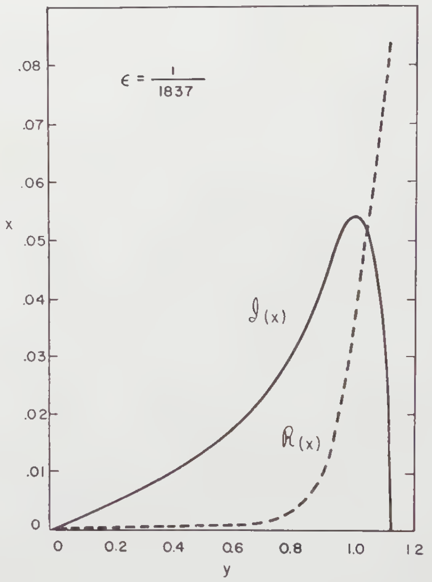

Fig. 9 The behavior of the real and imaginary part of the frequency of oscillation ($x = \omega/\omega_p$) as a function of the wave number of the disturbance ($k = \omega_p y/u$) for the unstable modes of oscillation.

Since Im dx/dy vanishes at the point in question, it follows that the denominator occurring in Eq. (125) must be real; we take

$$\frac{y}{x} - 1 = \alpha e^{+i\pi/3} \text{ at } y = y^*, \tag{126}$$

where α is a real positive number of order $\varepsilon^{-1/3}$. This choice of phase will be justified a posteriori. We use an asterisk to denote the value of any quantity correspondng to the maximum growth rate of instability. Substituting Eq. (126) into Eq. (113) and separating real and maginary parts, we obtain

$$y^2 = 1 + \frac{1}{\alpha^3(2+\alpha)}(1 + 2\alpha + 3\alpha^2) \tag{127}$$

and

$$\varepsilon = \frac{1}{\alpha^3}\frac{1+2\alpha}{2+\alpha}. \tag{128}$$

To the lowest significant order, Eq. (128) gives

$$\alpha = \left(\frac{2}{\varepsilon}\right)^{1/3}. \tag{129}$$

Substituting this value of α into Eq. (127), we obtain to the lowest significant order

$$y^* = 1 + \frac{3}{4}\left(\frac{\varepsilon}{2}\right)^{2/3}. \tag{130}$$

From Eqs. (126), (128), (129), and (130) we find that

$$x^* = \left(\frac{\varepsilon}{2}\right)^{1/3}\left[\frac{1}{2} - \left(\frac{\varepsilon}{2}\right)^{1/3} + \frac{3}{8}\left(\frac{\varepsilon}{2}\right)^{2/3} + \dots\right] + i\left(\frac{\varepsilon}{2}\right)^{1/3}\frac{3^{1/2}}{2}\left[1 + \frac{3}{4}\left(\frac{\varepsilon}{2}\right)^{2/3} + \dots\right]. \tag{131}$$

The maximum growth rate of instability, therefore corresponds to (for a hydrogen plasma)

$$\operatorname{Im} y^* = \operatorname{Im}\frac{\omega^*}{\omega_{pe}} \simeq \frac{3^{1/2}}{2}\left(\frac{\varepsilon}{2}\right)^{1/3} \simeq \frac{1}{18}. \tag{132}$$

Therefore the smallest *e*-folding time is 18 ω_{pe}^{-1}. For a plasma with a particle density of 10^{14} per cc, this time is of the order 10^{-10} seconds. These instabilities grow so rapidly that it seems plausible to view them as providing a mechanism for converting the organized energy, associated with the relative motion of ions and electrons, into the fluctuation energy of the oscillations. This provides an additional mechanism for the effective dissipation of the energy of organized plasma motion over and above collisions, and may consequently lead to enhanced resistivity of the plasma, compared to the corresponding value for a stable plasma. The behaviour of the real and imaginary parts of x^* as a function of ε is illustrated in Fig. 10.

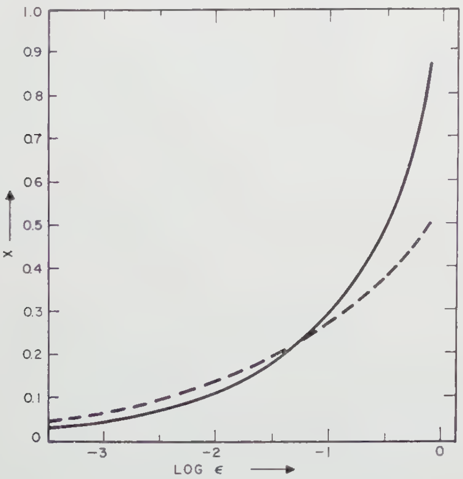

Fig. 10 The dependence of the maximum growth rate of instability as a function of ε is shown by the solid curve. The corresponding real part of the frequency is shown by the broken curve.

It may be emphasized here that these results are valid only so long as the phase velocities of the waves are much less than ion and electron thermal speeds. When this problem is examined from the standpoint of kinetic theory in part II, it will be seen that the thermal corrections transform certain stable regions in the $(u, \omega/k)$ plane into unstable regions, albeit with an exponentially small instability rate. Moreover, the region of the $(u, \omega/k)$ plane, corresponding roughly to the relative velocity **u** comparable to the ion thermal speed, is rendered stable.

The case of two interpenetrating beams of electrons (or ions) can be deduced also from the analysis given here by setting $\varepsilon = 1$ and by going over to a reference frame in which one of the beams is at rest. In this case one finds that the reciprocal of the smallest *e*-folding time is .866 ω_{pe} in contrast to (1/18) ω_{pe} as obtained earlier.

5. Plasma Carrying Current in a Magnetic Field

We now consider the case of a plasma, in which the electrons are moving with a constant velocity **u** relative to the ions, in the presence of an external magnetic field. We shall assume that the magnetic field which results from the steady state motion of the charges is negligible compared to the external constant magnetic field B_0. This is plausible if the fractional change in the magnetic field induced by these external currents is small over the wavelength of the perturbation in question. The equations for the problem at hand are (cf. Eqs. (58) and (60)):

$$(\omega^2 - c^2k^2)\,\mathbf{E} + c^2\mathbf{k}\,(\mathbf{k}\cdot\mathbf{E}) + 4\pi i\omega\sum eN\,\mathbf{v} + \frac{\mathbf{u}\,(\mathbf{k}\cdot\mathbf{v})}{(\omega - \mathbf{k}\cdot\mathbf{u})} = 0. \tag{133}$$

and

$$\mathbf{v} = \frac{e}{m}\frac{i}{\omega}\frac{1}{\omega_c^2 - (\omega - \mathbf{k}\cdot\mathbf{u})^2}\Big\{\boldsymbol{\omega}_c\,(\boldsymbol{\omega}_c\cdot\mathbf{E}) - (\omega - \mathbf{k}\cdot\mathbf{u})^2\,\mathbf{E} + \frac{1}{(\omega - \mathbf{k}\cdot\mathbf{u})}\left[\boldsymbol{\omega}_c\,(\boldsymbol{\omega}_c\cdot\mathbf{k}) - \mathbf{k}\,(\omega - \mathbf{k}\cdot\mathbf{u})^2\right](\mathbf{u}\cdot\mathbf{E}) - i\,(\omega - \mathbf{k}\cdot\mathbf{u})\,\boldsymbol{\omega}_c\times\mathbf{E} - i\,\boldsymbol{\omega}_c\times\mathbf{k}\,(\mathbf{u}\cdot\mathbf{E})\Big\}. \tag{134}$$

We shall first restrict ourselves to the case when $\mathbf{B}_0$, **u**, and **k** are along the *z*-axis. Then $\boldsymbol{\omega}_c \times \mathbf{k}$ vanishes and Eq. (134) reduces to

$$\mathbf{v} = \frac{e}{m}\frac{i}{\omega}\frac{1}{\omega_c^2 - (\omega - \mathbf{k}\cdot\mathbf{u})}\Big\{\left[\omega_c^2 - (\omega - ku)^2\right]E_z - (\omega - ku)^2\,(\mathbf{E} - \mathbf{n}\cdot\mathbf{E}) + \mathbf{k}\,\frac{\mathbf{u}\cdot\mathbf{E}}{(\omega - ku)}\left[\omega_c^2 - (\omega - ku)^2\right] - i\,(\omega - ku)\,\boldsymbol{\omega}_c\times\mathbf{E}\Big\}, \tag{135}$$

where **n** denotes a unit vector along the *z*-axis. From Eqs. (133) and (135) we obtain for longitudinal oscillations

$$\left[\omega^2 - \sum\omega_p^2\frac{\omega^2}{(\omega - ku)^2}\right]E_z = 0 \tag{136}$$

and for transverse oscillations

$$\left[\omega^2 - c^2k^2 - \sum_{+,-}\omega_p^2\frac{\omega - ku}{\omega - ku \mp \omega_c}\right]E_\pm = 0, \tag{137}$$

where

$$E_\pm = E_x \pm iE_y \tag{138}$$

represents the electric vector for the right and left handed circularly polarized waves respectively. The upper sign in the denominator in Eq. (137) corresponds to E_+, while the lower sign corresponds to E_-.

When E_z does not vanish, we obtain for longitudinal oscillations the dispersion relation

$$\sum \omega_p{}^2 \frac{1}{(\omega - ku)^2} = 1\,. \tag{139}$$

It is to be observed from this expression that the longitudinal oscillations are unaffected by the presence of an external magnetic field (cf. § 4). Consequently, our earlier remarks are applicable in this case as well. For propagation along any arbitrary direction the present discussion of longitudinal oscillations is also valid if one replaces k by $k_{||}$, the component of **k** along the magnetic field.

We shall now discuss the implications of the dispersion formula for the right handed circularly polarized wave only; a similar discussion applies to the left hand circularly polarized waves also and will not be given here. We have for the dispersion relation (cf. Eq. (137)):

$$\omega^2 - c^2 k^2 - \sum \omega_p{}^2 \frac{\omega - ku}{\omega - ku - \omega_c} = 0\,. \tag{140}$$

Denoting the stream velocity for the ions and electrons by u_i and u_e respectively, Eq. (140) takes the form

$$\omega^2 - c^2 k^2 - \omega_{pe}{}^2 \left(1 - \frac{\omega_e}{\omega - ku_e + \omega_e}\right) - \omega_{pi}{}^2 \left(1 + \frac{\omega_i}{\omega - ku_i - \omega_i}\right) = 0\,, \tag{141}$$

where, it may be recalled, ω_i and ω_e denote the ion and electron cyclotron frequencies respectively. Introducing the non-dimensional parameters

$$\left.\begin{aligned} x &= \frac{\omega}{\omega_{pe}}\,, y = \frac{ku}{\omega_{pe}}\,, \varepsilon = \frac{m}{M}\,, \\ z &= \frac{\omega_e}{\omega_{pe}}\,, \alpha = \frac{c}{u}\, y = \frac{ck}{\omega_{pe}} \end{aligned}\right\}\,, \tag{142}$$

we can write Eq. (141) as

$$F(x) = 1 + \varepsilon - x^2 + \alpha^2 - \varepsilon^2 \frac{z}{\varepsilon z - x + y_i} - \frac{z}{z + x - y_e} = 0\,. \tag{143}$$

We shall now discuss the implications of the relation (143).

Let us first consider the case when there is no stream. Eq. (143) then reduces to

$$F(x) = 1 + \varepsilon - x^2 + \alpha^2 - \varepsilon^2 \frac{z}{\varepsilon z - x} - \frac{z}{z + x} = 0\,. \tag{144}$$

The analysis of Eq. (144) is most readily effected by means of Fig. 11, where $F(x)$ is plotted schematically as a function of x. Observe that for large x, $F(x) \to -x^2$, $F(0) = \alpha^2$, and there are simple poles at $x = -z$ and $x = \varepsilon z$. Thus the curve $F(x)$ consists of three branches as indicated in Fig. 11 and there are clearly four real roots; consequently, all modes of oscillation are oscillatory and are *stable*.

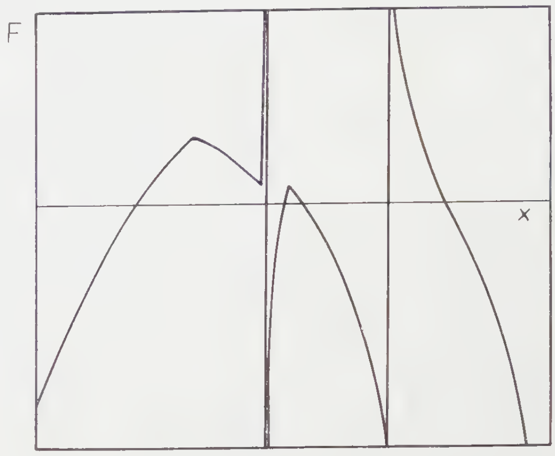

Fig. 11 A schematic plot of $F(x)$ given by Eq. (144) as a function of x. The singularities occur at the electron and ion cyclotron frequencies. The curve $F(x)$ intersects the x-axis at four points corresponding to four real roots.

5.1 Stream Instability (Transverse Oscillations)

We shall now consider the transverse oscillations for the case where we have two beams of fast electrons traversing the plasma. Suppose the fraction $1 - \beta$ of the electrons is moving with velocity $-\mathbf{u}_1$, and the fraction β is moving with velocity $\mathbf{u}_2$. The net current due to the electron streams is

$$\mathbf{J} = eN[-(1 - \beta)\,\mathbf{u}_1 + \beta \mathbf{u}_2]\,. \tag{145}$$

In order that the net current due to the electron streams vanish, we must demand that

$$(1 - \beta)\, u_1 = \beta u_2\,. \tag{146}$$

The corresponding results for the problem of one beam can be obtained immediately from these results by setting u_2 and β equal to zero. For the beam of relativistic electrons, the only modification in the dispersion relation is to replace the electron mass m by

$$m = \frac{m_0}{[1 - (u^2/c^2)]^{1/2}} = \gamma m_0\,, \tag{147}$$

where m_0 is the rest mass of the electrons. The dispersion relation now becomes

$$\omega^2 - c^2 k^2 - \omega_{pi}{}^2 \left(1 + \frac{\omega_i}{\omega - \omega_i}\right) - (1 - \beta) \frac{\omega_{pe}{}^2}{\gamma_1} \left[1 - \frac{\omega_e/\gamma_1}{\omega + (\omega_e/\gamma_1) + ku_1}\right] - \beta \frac{\omega_{pe}{}^2}{\gamma_2} \left[1 - \frac{\omega_e/\gamma_2}{\omega + (\omega_e/\gamma_2) - ku_2}\right] = 0\,, \tag{148}$$

where ω_{pe} now denotes the electron plasma frequency defined with respect to the electron rest mass, and, it will be recalled, that **k** and **u** are both along $\mathbf{B}_0$. Introducing the dimensionless variable given in Eq. (142), we can write Eq. (148) in the form

$$F(x) = (1-\beta)\frac{1}{\gamma_1} + \frac{\beta}{\gamma_2} + \varepsilon - x^2 + \alpha^2 - \varepsilon^2 \frac{z}{\varepsilon z - x}$$
$$-(1-\beta)\frac{1}{\gamma_1}\frac{z/\gamma_1}{x + y_1 + (z/\gamma_1)} - \frac{z\beta/\gamma_2^2}{x + y_2 + (z/\gamma_2)} = 0\,. \tag{149}$$

Firstly, we observe that if we have only one beam of electrons, Eq. (149) reduces to

$$F(x) = 1 + \varepsilon - x^2 + \alpha^2 - \varepsilon^2 \frac{z}{\varepsilon z - x}$$
$$- \frac{1}{\gamma_1^2}\frac{z}{x - y_1 + (z/\gamma_1)} = 0\,. \tag{150}$$

It turns out that Eq. (150) has real roots, corresponding to oscillatory motions, for all values of the parameters of physical interest.

We now consider the case of two beams of electrons. The first point which is to be observed is that the effect of adding another beam is to introduce a new pole as shown schematically in Fig. 12. As u_2 increases from its zero value, the additional root moves to the left in Fig. 12.

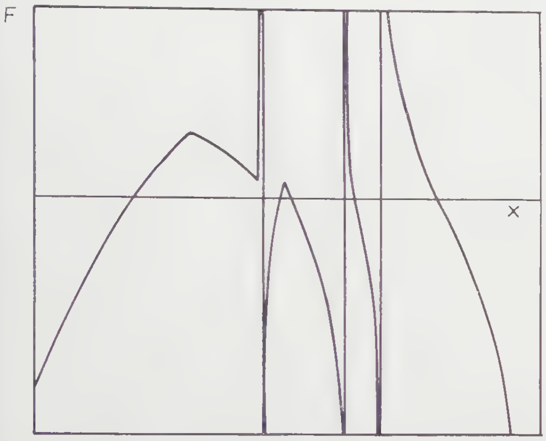

Fig. 12 A schematic plot of $F(x)$ given by Eq. (150) as a function of x. Note the additional branch of the curve introduced by the presence of interpenetrating beams of fast electrons. The curve $F(x)$ intersects the x-axis five times corresponding to all real roots.

When u_2 is so large that the gap introduced in the curve of Fig. 12 straddles the x-axis, two real roots are lost and a pair of complex conjugate roots are acquired. One of these must have a positive imaginary part and corresponds to instability. This is illustrated in Fig. 13.

When instability does occur it has a simple physical interpretation; namely, the condition that the gap straddles the x-axis is that

$$x + y_2 + (z/\gamma_2) \simeq 0\,. \tag{151}$$

This can be interpreted as saying that, in its rest frame, the electron beam sees the circularly polarized wave as having a frequency equal to the electron cyclotron frequency, and the in-phase rotating component of the electric field is resonant with the particle motion. Since instability occurs only when the gap straddles the x-axis, it is clear that for this to occur the new pole must lie between the electron and ion cyclotron frequencies, i.e., between $-z$ and εz.

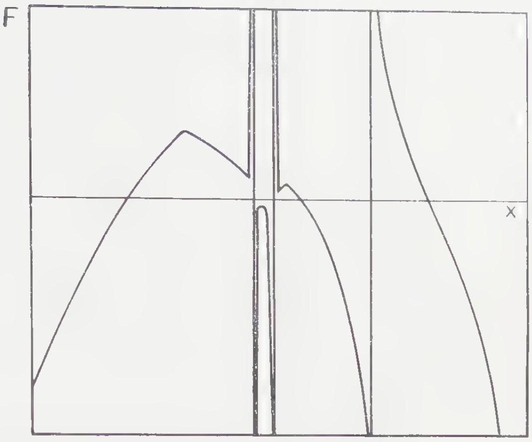

Fig. 13 A schematic plot of $F(x)$ given by Eq. (150) as a function of x for the case where the interpenetrating beams have velocities corresponding to unstable modes of oscillation. Observe that $F(x)$ now intersects the x-axis only three times, corresponding to three real roots and two complex roots.

5.11 Oscillations much below the Ion Cyclotron Frequency

We now consider the consequences on the hydromagnetic waves given by Eq. (78) of the introduction of beams of fast electrons such that the small gap straddles the real axis near the unperturbed root (denoted by subscript 1):

$$x_1 = -\alpha\left(1 + \frac{1}{\varepsilon z^2}\right)^{-1/2}. \tag{152}$$

Then for the case contemplated

$$F(x) = \varepsilon - x^2 + \alpha^2 - \varepsilon\left(1 - \frac{x}{\varepsilon z}\right)^{-1} + \frac{1}{\gamma_1}(1-\beta)$$
$$+ \frac{1}{\gamma_2}\beta - (1-\beta)\frac{1}{\gamma_1}\left(1 + \frac{x + y_1}{z/\gamma_1}\right)^{-1}$$
$$- \frac{\beta}{\gamma_2^2}\frac{z}{x - y_2 + (z/\gamma_2)} = 0 \tag{153}$$

or

$$F(x) = -x^2 + \alpha^2 - \left(\frac{x}{z} + \frac{x^2}{\varepsilon z^2} + \dots\right)$$
$$+ (1-\beta)\frac{1}{\gamma_1}\left(\frac{x + y_1}{z/\gamma_1} + \dots\right)$$
$$+ \frac{\beta}{\gamma_2}\left[1 - \frac{z/\gamma_2}{x - y_2 + (z/\gamma_2)}\right] = 0, \tag{154}$$

$$= -x^2\left(1 + \frac{1}{\varepsilon z^2}\right) + \alpha^2 - \beta\frac{x - y_2}{z}$$
$$+ \frac{\beta}{\gamma_2}\left[1 - \frac{z/\gamma_2}{x - y_2 + (z/\gamma_2)}\right] = 0, \tag{155}$$

where in writing Eq. (155) we have used the fact that

$$(1-\beta)y_1 = \beta y_2\,. \tag{156}$$

Making use of Eq. (152), we can write Eq. (155) as

$$F(x) = (x_1^2 - x^2)\left(1 + \frac{1}{\varepsilon z^2}\right) + \beta\left(\frac{1}{\gamma_2} - \frac{x}{z} + \frac{y_2}{z}\right)$$
$$- \frac{\beta z}{\gamma_2^2}\frac{1}{x - y_2 + (z/\gamma_2)} = 0\,. \tag{157}$$

For the case under consideration $x/\varepsilon z \ll 1$ and, further, since we are interested in the case when $x \sim x_1$, we may set

$$x_1^2 - x^2 \simeq -2x_1(x - x_1). \tag{158}$$

Eq. (151) then reduces to (assuming $x/z \ll 1$):

$$F(x) = -\left(1 + \frac{1}{\varepsilon z^2}\right) 2x_1(x - x_1) + \beta\left(\frac{1}{\gamma_2} + \frac{y_2}{z}\right) - \frac{\beta z}{\gamma_2^2\,[x - y_2 + (z/\gamma_2)]} = 0. \tag{159}$$

Letting

$$\left.\begin{aligned} p &= \left(1 + \frac{1}{\varepsilon z^2}\right) 2x_1, \\ q &= \beta\left(\frac{1}{\gamma_2} + \frac{y_2}{z}\right), \\ r &= -\frac{\beta z}{\gamma_2^2}, \end{aligned}\right\} \tag{160}$$

Eq. (159) then leads to the following quadratic equation for x:

$$x^2 + x\left(-x_1 + \frac{z}{\gamma_2} - y_2 - \frac{q}{p}\right) + \left(x_1 + \frac{q}{p}\right)\left(y_2 - \frac{z}{\gamma_2}\right) - \frac{r}{p} = 0. \tag{161}$$

The roots of this equation are given by

$$x = +\frac{1}{2}\left(x_1 - \frac{z}{\gamma_2} + y_2 + \frac{q}{p}\right) \pm \frac{1}{2}\left[\left(x_1 + \frac{q}{p} - y_2 + \frac{z}{\gamma_2}\right)^2 + \frac{4r}{p}\right]^{1/2}. \tag{162}$$

Explicitly, we may write this as

$$\omega = +\frac{1}{2}\left[\omega_1 - \frac{\omega_e}{\gamma_2} + ku_2 + \beta\frac{[(1/\gamma_2) - (ku_2/\omega^{\circ})]\,\omega^2{}_{pe}}{2|\omega_1|\,[1 + (4\pi NMc^2/B_0^2)]}\right]$$
$$\pm\frac{1}{2}\left[\left\{\omega_1 + \frac{\omega_e}{\gamma_2} - ku_2 + \beta\frac{[(1/\gamma_2) - (ku_2/\omega_c)]\,\omega^2{}_{pe}}{2|\omega_1|\,[1 + (4\pi NMc^2/B_0^2)]}\right\}^2 - \frac{2\beta\omega_{pe}^2\omega_e/\gamma_2^2}{|\omega_1|\,[1 + (4\pi NMc^2/B_0^2)]}\right]^{1/2} \tag{163}$$

It is clear from this expression that the maximum instability rate occurs when the term in the braces vanishes; this is indeed the approximate criterion given in Eq. (151). The spread in velocity of the electron beam about that given in Eq. (151) for which there is instability is given by (cf. Eq. (163)):

$$k\Delta u_2 = \frac{2\beta\omega_{pe}^2\omega_e/\gamma_2^2}{|\omega_2|\,[1 + (4\pi NMc^2/B_0^2)]}. \tag{164}$$

5.12 Oscillations near the Ion Cyclotron Frequency

Let us now consider the case when the physical parameters are such that in the absence of the beam the interesting root is given by Eq. (81) — namely, near the ion cyclotron frequency. For the case under consideration, $x \ll z$, and we shall assume that y_1 is of order x. Denoting the root in question by the subscript 2, we have (cf. Eq. (81))

$$x_2 = \varepsilon z\left(1 + \frac{\varepsilon}{\varepsilon^2 z^2 - \varepsilon - \alpha^2}\right). \tag{165}$$

Equation (149) can then be written in the form

$$F(x) - \varepsilon^2 z\left(\frac{1}{\varepsilon z - x_2} - \frac{1}{\varepsilon z - x}\right) + \frac{\beta}{\gamma_2} - \frac{\beta}{\gamma_2^2}\frac{z}{x - y_2 + (z/\gamma_2)} = 0, \tag{166}$$

$$\simeq -\varepsilon^2 z\,\frac{x - x_2}{(\varepsilon z - x_2)^2} + \frac{\beta}{\gamma_2} - \frac{\beta}{\gamma_2^2}\frac{z}{x - y_2 + (z/\gamma_2)} = 0. \tag{167}$$

Setting

$$p_1 = \frac{\varepsilon^2 z}{(\varepsilon z - x_2)^2},\ q_1 = \frac{\beta}{\gamma_2},\ r_1 = \frac{\beta z}{\gamma_2^2}, \tag{168}$$

Eq. (167) leads to the quadratic

$$x^2 + x\left(-x_2 - y_2 + \frac{z}{\gamma_2} - \frac{q_1}{p_1}\right) + \left(y_2 - \frac{z}{\gamma_2}\right)\left(x_2 + \frac{q_1}{p_1}\right) + \frac{r_1}{p_1} = 0. \tag{169}$$

The roots of this equation are given by

$$x = +\frac{1}{2}\left(+x_2 + y_2 - \frac{z}{\gamma_2} + \frac{q_1}{p_1}\right) \pm \frac{1}{2}\left[\left(x_2 - y_2 + \frac{z}{\gamma_2} + \frac{q_1}{p_1}\right)^2 - \frac{4r_1}{p}\right]^2. \tag{170}$$

The maximum growth rate of instability occurs when

$$x_2 - y_2 + \frac{z}{\gamma_2} + \frac{q_1}{p_1} = 0, \tag{171}$$

which is again similar to the approximate criterion stated in Eq. (151). The spread in velocity about that given by Eq. (151) for which instability does occur is given by

$$k\Delta u_2 = \frac{4r_1}{p_1} = \frac{4\beta z}{\gamma_2}\frac{(\varepsilon z - x_2)^2}{\varepsilon^2 z}. \tag{172}$$

The instabilities due to the presence of streams of electrons (or ions) may be responsible for microwave noise in the laboratory plasmas and the radio emission from astronomical objects.

6. SUMMARY OF SECTIONS III 2 — III 5

We shall present here a summary of the dispersion relations for the various modes of oscillation of a plasma.

a) Static Plasma in the Absence of a Magnetic Field

(i) Longitudinal electron oscillations ($\omega^2 \gg k^2\,\Theta/m$)

$$\omega^2 = \omega_{pe}^2 + 3\frac{\Theta}{m}k^2 \tag{173}$$

(ii) Transverse electron oscillations ($\omega^2 \gg k^2\,\Theta/m$)

$$\omega^2 = \omega_{pe}^2 + c^2k^2 + k^2\frac{\Theta}{m}\frac{\omega_{pe}^2}{\omega_{pe}^2 + c^2k^2} \tag{174}$$

(iii) Ion oscillations ($\Theta_i \ll \Theta_e$)

$$\omega^2 = \frac{1}{1 + (k\lambda_D)^2}\omega_{pi}^2 + 3k^2\frac{\Theta_i}{m} \tag{175}$$

b) Static Plasma in a Magnetic Field, $\mathbf{k} \parallel \mathbf{B}_0$

(iv) High frequency oscillations $\omega \gg \omega_{e,i}$, $\omega_e^2 \ll \omega_{pe}^2 + c^2k^2$

$$\omega^2 = c^2 k^2 + \omega_{pe}^2 \left[1 - \frac{\omega_e}{(\omega_{pe}^2 + c^2 k^2)^{1/2}}\right] \qquad (176)$$

(v) Oscillations near electron cyclotron frequency $(\omega_e^2 > \omega_p^2 + c^2 k^2)$

$$\omega = -\omega_e \left[1 - \frac{\omega_{pe}^2}{\omega_e^2 - (c^2 k^2 + \omega_{pe}^2)}\right] \qquad (177)$$

(vi) Oscillations much below the ion cyclotron frequency

$$\omega^2 = \frac{c^2 k^2}{1 + (c^2/A^2)} \qquad (178)$$

(vii) Oscillations near the ion cyclotron frequency $(\omega_i^2 \gg \omega_{pi}^2 + c^2 k^2)$

$$\omega = \omega_i \left[1 + \frac{\omega_{pi}^2}{\omega_i^2 - (c^2 k^2 + \omega_{pi}^2)}\right] \qquad (179)$$

c) Static Plasma in a Magnetic Field, $\mathbf{k} \perp \mathbf{B}$

(viii) Lower hybrid frequency $(\omega \ll ck)$, $\omega_i < \omega < \omega_e < \omega_{pe}$

$$\omega = (\omega_e \omega_i)^{1/2} \qquad (180)$$

(ix) Upper hybrid frequency, $\omega > \omega_i, \omega > \omega_{pi}$

$$\omega = (\omega_e^2 + \omega_{pe}^2)^{1/2} \qquad (181)$$

d) Stream Instabilities

1. Longitudinal oscillations, in the presence of a stream of electrons with constant velocity **u**, are unstable to all perturbations (along **u**) having wavelengths greater than

$$\lambda_c = 2\pi \frac{u}{\omega_{pe}} \left(1 + \frac{3}{2} \frac{m}{M}\right)^{-1}. \qquad (182)$$

Smallest e-folding time $= 18\, \omega_{pe}^{-1}$.
This instability is unaffected by the presence of a magnetic field along **u**.

2. Transverse oscillations, in the presence of two streams of relativistic electrons, are unstable near the Alfvén frequency and the ion cyclotron frequency, **k** and **u** being assumed to be along **B**.

7. HYDROMAGNETIC WAVES FROM CHEW, GOLDBERGER, AND LOW THEORY

We have shown earlier that, for a low density plasma n a strong magnetic field, the kinetic equations for the lasma reduce to equations for a single fluid if we eglect the heat flow along the lines of force and/or ssume the low temperature approximation. We shall ow discuss the propagation of hydromagnetic waves sing this formalism.

Consider, then, a spatially homogeneous plasma in tatic equilibrium in a uniform magnetic field, which we hall assume to be along the z-axis of a Cartesian system f co-ordinates, with the pressure tensor given by

$$\mathbf{P} = p_\perp \mathbf{I} + (p_\| - p_\perp)\, \mathbf{n}\,\mathbf{n}, \qquad (183)$$

vhere **n** denotes a unit vector along the field lines. In the erturbed state, let the various quantities be denoted by

$$\mathbf{v}, \varrho + \delta\varrho, p_\perp + \delta p_\perp, p_\| + \delta p_\|, \mathbf{B} + \delta\mathbf{B}, \qquad (184)$$

The equations governing the perturbed quantities are obtained by linearizing Eqs. (II-135)—(II-140). These are

$$\varrho \frac{\partial \mathbf{v}}{\partial t} = -\nabla \cdot \delta\mathbf{P} + \frac{1}{4\pi} (\nabla \times \delta\mathbf{B}) \times \mathbf{B}, \qquad (185)$$

$$\frac{\partial}{\partial t} \delta\varrho = -\varrho \nabla \cdot \mathbf{v}, \qquad (186)$$

$$\frac{\partial}{\partial t} \delta\mathbf{B} = \nabla \times (\mathbf{v} \times \mathbf{B}), \qquad (187)$$

$$\delta p_\perp = p_\perp \frac{\delta B}{B} + p_\| \frac{\delta\varrho}{\varrho}, \qquad (188)$$

and

$$\delta p_\| = -2p_\| \frac{\delta B}{B} + 3p_\| \frac{\delta\varrho}{\varrho}, \qquad (189)$$

where $\delta B = \mathbf{B} \cdot \delta\mathbf{B}/B$. Introducing the Lagrangian displacement $\boldsymbol{\xi}, \mathbf{v} = \partial\boldsymbol{\xi}/\partial t$, Eqs. (186) and (187) can be readily integrated to give

$$\delta\varrho = -\varrho \nabla \cdot \xi \qquad (190)$$

and

$$\delta\mathbf{B} = \nabla \times (\xi \times \mathbf{B}), \qquad (191)$$

where the constant of integration, by definition, has been set equal to zero. It follows readily from Eq. (191) that

$$\delta B = -\nabla_\perp \cdot \xi_\perp, \qquad (192)$$

where we use a subscript $\perp$ to denote the component of a quantity in the plane transverse to **B**. Thus

$$\nabla_\perp = \nabla - \mathbf{n}\mathbf{n} \cdot \nabla. \qquad (193)$$

The component of a quantity along **B** will be denoted by the subscript $\|$. We shall assume that all the perturbed quantities behave like $\exp i(-\omega t + \mathbf{k} \cdot \mathbf{r})$. Using the foregoing results, Eqs. (188) and (189) reduce to

$$\delta p_\perp = -p_\perp (2\nabla_\perp \cdot \xi_\perp + ik_\| \xi_\|) \qquad (194)$$

and

$$\delta \nabla p_\| = -p_\| (\nabla_\perp \cdot \xi_\perp + 3ik_\| \xi). \qquad (195)$$

From the form of **P**, it follows that

$$\nabla \cdot \delta\mathbf{P} = \nabla_\perp \delta p_\perp + i(p_\| - p_\perp) k_\| \delta\mathbf{n} + \mathbf{n}(p_\| - p_\perp) \nabla \cdot \delta\mathbf{n} + ik_\| \delta p_\|]. \qquad (196)$$

The change $\delta\mathbf{n}$ in the unit vector along the lines of force is obtained readily from Eq. (191):

$$\delta\mathbf{n} = \mathbf{n} \cdot \nabla \xi_\perp + [(\mathbf{n} \cdot \nabla)\, \mathbf{n}\mathbf{n}] \cdot \xi. \qquad (197)$$

Using the relation (197) into (196), we obtain

$$\nabla \cdot \delta\mathbf{P} = \nabla_\perp \delta p_\perp + (p_\perp - p_\|) k_\|^2 \xi_\perp + ik_\| [(p_\| - p_\perp) \nabla_\perp \cdot \xi_\perp + p_\|]\, \mathbf{n}. \qquad (198)$$

From Eq. (191) it follows readily that

$$(\nabla \times \delta\mathbf{B}) \times \mathbf{B} = B^2 (\nabla_\perp \nabla_\perp \cdot \xi_\perp - k_\|^2 \xi_\perp). \qquad (199)$$

Using the foregoing results the equation of motion (185) splits up into the two equations

$$\left[\varrho\omega^2 + k_\|^2 \left(p_\| - p_\perp - \frac{B^2}{4\pi}\right)\right] \xi_\perp = \mathbf{k}_\perp \left[2(\mathbf{k}_\perp \cdot \xi_\perp) \left(p_\perp + \frac{B^2}{8\pi}\right) + k_\| p_\perp \xi_\|\right], \qquad (200)$$

and

$$(\varrho\omega^2 - 3p_\parallel k_\parallel^2)\,\xi_\parallel = k_\parallel p_\perp (\mathbf{k}_\perp \cdot \xi_\perp)\,. \tag{201}$$

On substituting for $\xi_\parallel$ in accordance with Eq. (201) into (200) and forming the scalar product with $\mathbf{k}_\perp$, we obtain the dispersion relation

$$\varrho\omega^2 + k_\parallel\left(p_\parallel - p_\perp - \frac{B^2}{4\pi}\right) - 2k_\perp^2\left(p_\perp + \frac{B^2}{8\pi}\right) = k_\parallel^2 p_\perp^2 \frac{k_\perp^2}{\varrho\omega^2 - 3p_\parallel k_\parallel^2} \tag{202}$$

If θ denotes the angle which the propagation vector makes with the z-axis,

$$k_\parallel = k\cos\theta\,, \qquad k_\perp = k\sin\theta\,. \tag{203}$$

After some simplifications, Eq. (202) reduces to the following quadratic equation for ω^2:

$$\omega^4 - \frac{\omega^2}{\varrho}\,k^2\left(\frac{B^2}{4\pi} + p_\perp + 2p_\parallel \cos^2\theta + p_\perp \sin^2\theta\right) + 3\frac{k^4}{\varrho^2}p_\parallel \cos^2\theta\left[\frac{B^2}{4\pi} + p_\perp(1+\sin^2\theta) - p_\parallel \cos^2\theta\right] - \frac{k^4}{\varrho^2}p_\perp \sin^2\theta\cos^2\theta = 0\,. \tag{204}$$

The roots of this equation are given by

$$\omega^2 = \frac{k^2}{2\varrho}\left(\frac{B^2}{4\pi} + p_\perp + 2p_\parallel \cos^2\theta + p_\perp \sin^2\theta\right) + \frac{k^2}{2\varrho}\left\{\left[\frac{B^2}{4\pi} + p_\perp(1+\sin^2\theta) - 4p_\parallel \cos^2\theta\right]^2 + 4p_\perp^2 \sin^2\theta\cos^2\theta\right\}^{1/2} \tag{205}$$

For propagation along the lines of force, $\theta = 0$ and Eq. (205) leads to the following two modes of oscillation (distinguished by the subscripts 1 and 2):

$$\omega_1^2 = \frac{k^2}{\varrho}\left(\frac{B^2}{4\pi} + p_\perp - p_\parallel\right) \tag{206}$$

and

$$\omega_2^2 = \frac{3k^2}{\varrho}p_\parallel. \tag{207}$$

It is clear that the mode of oscillation corresponding to Eq. (206) is unstable if

$$p_\parallel > \frac{B^2}{4\pi} + p_\perp. \tag{208}$$

That is, if the distribution function for the ions and/or electrons is strongly peaked along the lines of force, the hydromagnetic waves propagating along the field lines are unstable.

For propagation transverse to the field lines, $\theta = \pi/2$ and we obtain

$$\omega^2 = \frac{2k^2}{\varrho}\left(\frac{B^2}{8\pi} + p_\perp\right). \tag{209}$$

The corresponding phase velocity of the wave is given by

$$v_p = \frac{\omega}{k} = \left(\frac{2P}{\varrho}\right)^{1/2}, \tag{210}$$

where $P = p_\perp + (B^2/8\pi)$ denotes the total transverse pressure.

For any arbitrary direction of propagation there exist unstable modes of oscillation for all θ's less than θ_c as determined by the equation

$$\frac{B^2}{4\pi} + p_\perp(1+\sin^2\theta_c) = \frac{p_\perp}{3p_\parallel}\sin^2\theta_c + p_\parallel \cos^2\theta_c. \tag{211}$$

As an example, if we take $p_\perp = B^2/8\pi$ and $p_\parallel = B^2/2\pi$, we find from Eq. (211) that $\theta_c = 26^0\,45'$.

It will be shown in part II of this survey that in the presence of a magnetic field, whenever the distribution function is anisotropic (i.e., $P_\parallel < P_\perp$ or $P_\perp < P_\parallel$), we have instabilities, although their e-folding times tend to be rather small.

8. HYDROMAGNETIC WAVES: GENERAL THEORY

In our previous simplified discussion of hydromagnetic waves we neglected the effects of the finite temperature, viscosity, electrical conductivity, thermal conductivity, and other transport co-efficients of the plasma. These shall now be taken into account in order to exhibit the coupling of the hydromagnetic and sound waves, and to discuss the damping of these waves due to dissipative effects. However, for the sake of simplicity, we shall assume that the ion gyration frequency is much smaller than the effective collision frequency so that all the transport co-efficients may be taken to be isotropic. Eqs. (II-94)—(II-97) then form the basis of our discussion.

In the equilibrium state $\mathbf{U} = 0$ and $\mathbf{B} = B_0\hat{e}_z$, where B_0 is constant and p and ϱ are uniform. For small departures from the equilibrium state let the various quantities be denoted by

$$\mathbf{v}\,, p + p_1, \quad \varrho + \varrho_1, \quad \text{and } \mathbf{B}_0 + \mathbf{B}_1. \tag{212}$$

The equations governing the small departures are obtained by linearizing the equations of motion (II-94)—(II-97). These are:

$$\frac{\partial \varrho_1}{\partial t} + \varrho\nabla\cdot\mathbf{v} = 0\,, \tag{213}$$

$$\varrho\frac{\partial \mathbf{v}}{\partial t} = -\nabla p_1 + \frac{1}{4\pi}(\nabla\times\mathbf{B}_1)\times\mathbf{B}_0 + \mu\nabla^2\mathbf{v} + \frac{1}{3}\mu\nabla\nabla\cdot\mathbf{v}, \tag{214}$$

$$\frac{\partial \mathbf{B}_1}{\partial t} = \nabla\times(\mathbf{v}\times\mathbf{B}_0) + \frac{c^2}{4\pi\sigma_1}\nabla^2\mathbf{B}_1\,, \tag{215}$$

and

$$\frac{\partial p_1}{\partial t} = s^2\frac{\partial \varrho_1}{\partial t} + \frac{2}{3}\nabla\cdot(\lambda_1\nabla\Theta_1 + \nu_1\mathbf{D}_1)\,, \tag{216}$$

where $s = (\gamma p/\varrho)^{1/2}$ denotes the sound speed and $\mathbf{D}_1$ is obtained from the linearized form of Eq. (II-89). Thus

$$\mathbf{D}_1 = \frac{1}{\sigma_1}\left(\frac{c}{4\pi}\nabla\times\mathbf{B}_1 - \varkappa\nabla\Theta_1\right). \tag{217}$$

The temperature and the pressure are connected by the relation $\Theta = Mp/\varrho$; the linearized form of this gives

$$\Theta_1 = \frac{M}{\varrho}\left(p_1 - p\frac{\varrho_1}{\varrho}\right). \tag{218}$$

Introducing $\boldsymbol{\xi}$, the Lagrangian displacement of an lement of the fluid, and assuming that all quantities ehave like exp $i\,(-\omega t + \mathbf{k}\cdot\mathbf{r})$, after some straightorward reductions Eqs. (213), (215), and (216) lead to

$$\varrho_1 = -\varrho\nabla\cdot\boldsymbol{\xi}\,, \tag{219}$$

$$\mathbf{B}_1 = \alpha\nabla\times(\boldsymbol{\xi}\times\mathbf{B}_0)\,, \tag{220}$$

nd

$$p_1 = \varphi s^2 \varrho_1\,, \tag{221}$$

vhere

$$\frac{1}{\alpha} = 1 + \frac{i k^2 c^2}{4\pi\sigma_1\omega} \tag{222}$$

nd

$$\varphi = \frac{1 + \frac{2}{3} i\,[\lambda_1 - (\nu_1 \varkappa_1/\sigma_1)][k^2 M/(\gamma\,\omega\,\varrho)]}{1 + \frac{2}{3} i\,[\lambda_1 - (\nu_1 \varkappa_1/\sigma_1)][k^2 M/(\varrho\,\omega)]}\,. \tag{223}$$

t must be emphasized here that in writing the foregoing xpressions we have assumed the various transport o-efficients to be isotropic. However, this may not be rue, in general, for oscillating electromagnetic fields.

Using the foregoing results in the equation of motion 214), we obtain

$$\left(\omega^2 + i\omega\frac{\mu}{\varrho}k^2\right)\boldsymbol{\xi} = \left(s^2\varphi - \frac{1}{3}i\omega\frac{\mu}{\varrho}\right)\mathbf{k}\,\mathbf{k}\cdot\boldsymbol{\xi} + \alpha A^2\{\mathbf{k}\times[\mathbf{k}\times(\boldsymbol{\xi}\times\mathbf{n})]\}\times\mathbf{n}\,, \tag{224}$$

vhere $\mathbf{n}$ is a unit vector along $\mathbf{B}_0$ and A is the usual Alfvén speed. Assuming that $\mathbf{k}$ lies in the (xz) plane naking an angle θ with the z-axis, Eq. (224) leads to the ollowing equations for the different components of $\boldsymbol{\xi}$:

$$\left(\omega^2 - \alpha A^2 k^2 \cos^2\theta + \frac{i\mu}{\varrho}k^2\omega\right)\xi_y = 0\,, \tag{225}$$

$$\left[\omega^2 - \alpha A^2 k^2 + i\frac{\mu}{\varrho}k^2\omega - \left(s^2\varphi - \frac{i\mu}{3\varrho}\omega\right)k^2\sin^2\theta\right]\xi_x - \left(\varphi s^2 - i\frac{\omega\mu}{3\varrho}\right)k^2\sin\theta\cos\theta\,\xi_z = 0\,, \tag{226}$$

ınd

$$-\left(\varphi s^2 - \frac{i\omega\mu}{3\varrho}\right)k^2\sin\theta\cos\theta\,\xi_x + \left[\omega^2 + \frac{i\mu}{\varrho}\omega k^2 - \left(\varphi s^2 - \frac{i\mu}{3\varrho}\omega\right)k^2\cos^2\theta\right]\xi_z = 0\,. \tag{227}$$

8.1 *Absence of dissipation*

Let us first consider the limiting case of an ideal ydromagnetic fluid for which all the transport co-effiients vanish. In this limiting situation $\alpha = 1$, $\varphi = 1$, nd $\mu = 0$. Eq. (225) then leads to

$$\omega = \pm A k \cos\theta\,; \tag{228}$$

hese modes of oscillation remain unaffected by the finite lasma pressure. For Eqs. (226) and (227) to have a on-trivial solution, we must have (setting $\alpha = 1$, $= 1, \mu = 0$):

$$\begin{vmatrix} \omega^2 - A^2k^2 - s^2k^2\sin^2\theta & -s^2k^2\sin\theta\cos\theta \\ -s^2k^2\sin\theta\cos\theta & \omega^2 - s^2k^2\cos^2\theta \end{vmatrix} = 0\,. \tag{229}$$

This leads to the following quadratic for ω^2:

$$\omega^4 - \omega^2(A^2 + s^2)k^2 + A^2 s^2 k^4 \cos^2\theta = 0\,. \tag{230}$$

The roots of this equation are given by

$$\omega^2 = \frac{1}{2}\left\{(A^2+s^2)k^2 \pm \frac{1}{2}k^2(A^2+s^2)\left(1 - \frac{4s^2A^2\cos^2\theta}{(s^2+A^2)^2}\right)^{1/2}\right\}. \tag{231}$$

In the case of a strong magnetic field, $s/A \ll 1$, and Eq. (231) leads to the following roots (distinguished by the subscripts 1 and 2):

$$\omega_1^2 = k^2(A^2 + s^2\sin^2\theta) \tag{232}$$

and

$$\omega_2^2 = k^2 s^2 \cos^2\theta\,. \tag{233}$$

In the case of a weak magnetic field, $A/s \ll 1$, and Eq. (231) leads to

$$\omega_1^2 = k^2(s^2 + A^2\sin^2\theta) \tag{234}$$

and

$$\omega_2^2 = k^2 A^2 \cos^2\theta\,. \tag{235}$$

The principal result which stems from including the finite plasma pressure is that the hydromagnetic waves remain unaffected only if the motion of the material (i.e., $\boldsymbol{\xi}$) is normal to the plane containing the magnetic field and the propagation vector; otherwise the disturbance breaks up into coupled hydromagnetic waves and sound waves. The chief application of this result is to waves originating in the surface layers of a sunspot where A may be comparable to s. These remarks may be of some significance also in laboratory plasma diagnostics.

That the ω^2 are real in the case of ideal hydromagnetics also follows from the self adjointness of the equations of motion which may, in general, be written in the form

$$-\omega^2\varrho\,\boldsymbol{\xi} = \mathbf{F}(\boldsymbol{\xi})\,, \tag{236}$$

where $\mathbf{F}$ is a linear operator acting on $\boldsymbol{\xi}$. The self adjointness of $\mathbf{F}$ immediately leads to the conclusion that ω^2 is real. Consequently, the criterion for stability is that there be no negative characteristic value ω^2. Also, it can be shown easily that a necessary and sufficient condition for this circumstance is that the potential energy

$$\frac{1}{2}\int d^3\mathbf{x}\;\boldsymbol{\xi}\cdot\mathbf{F}(\boldsymbol{\xi}) \tag{236a}$$

always be positive. This latter condition is a statement of the energy principle governing the stability of ideal hydromagnetic system. This criterion, while derived on the assumption of collision-dominated plasma, can be demonstrated to give a lower bound on stability as compared with the conclusions of the theories based on the collisionless Boltzmann equation in the limit of a strong magnetic field.

8.2 *Effects of dissipation*

We now consider the effects of dissipation on the various modes of oscillation discussed in the preceding section. Eq. (225) gives

$$\omega^2 = A^2 k^2 \cos^2\theta \left(1 + i\frac{c^2 k^2}{4\pi\sigma_1\omega}\right)^{-1} - \frac{i\mu}{\varrho} k^2 \omega. \tag{237}$$

If $c^2k^2/4\pi\sigma_1\omega$ and $\mu\omega/\varrho A^2$ are both much less than unity, to the lowest significant order Eq. (237) yields

$$\omega = Ak\cos\theta - \frac{1}{2} ik^2 \left(\frac{c^2}{4\pi\sigma_1} + \frac{\mu}{\varrho}\right). \tag{238}$$

We now consider Eqs. (226) and (227). In order that these admit a non-trivial solution, Eq. (239) see bottom of this page.

On the assumption that $c^2k^2/4\pi\sigma_1\omega$, $\mu k^2/\varrho\omega$, and $[\lambda_1 - (\nu_1\varkappa_1/\sigma_1)]\,[k^2 M/\omega\varrho]$ are all much less than unity, we obtain in the strong field case ($s/A \ll 1$):

$$\omega_1{}^2 = k^2 A^2 \left\{1 + \frac{s^2}{A^2}\sin^2\theta - \frac{ik}{A}\left[\frac{c^2}{4\pi\sigma_1} + \frac{\mu}{\varrho}(1+\sin^2\theta) + \frac{2}{3}\frac{s^2}{A^2}\left(\lambda_1 - \frac{\nu_1\varkappa_1}{\sigma_1}\right)\frac{M}{\varrho}\frac{\gamma-1}{\gamma}\right]\right\} \tag{240}$$

and

$$\omega_2{}^2 = k^2 \left\{s^2\cos^2\theta - ikA\left[\frac{\mu}{\varrho}(1+\cos^2\theta) + \frac{2}{3}\frac{s^2}{A^2}\left(\lambda_1 - \frac{\nu_1\varkappa_1}{\sigma_1}\right)\frac{M}{\varrho}\frac{\gamma-1}{\gamma}\right]\right\}. \tag{241}$$

Under the same approximations for the weak field case we obtain

$$\omega_1{}^2 = k^2 s^2 \left\{1 + \frac{A^2}{s^2}\sin^2\theta - \frac{ik}{s}\left[\frac{4}{3}\frac{\mu}{\varrho} + \frac{2}{3}\frac{s^2}{A^2}\left(\lambda_1 - \nu_1\frac{\varkappa_1}{\sigma_1}\right)\frac{M}{\varrho}\frac{\gamma-1}{\gamma}\right]\right\} \tag{242}$$

and

$$\omega_2{}^2 = k^2\left(A^2\cos^2\theta + \frac{i\mu}{\varrho} sk\right). \tag{243}$$

It is clear from Eqs. (240) and (241) that the damping of the hydromagnetic waves is

$$\exp\left\{-\frac{k^2}{2}\left[\frac{c^2}{4\pi\sigma_1} + \frac{\mu}{\varrho} + \frac{2}{3}\frac{s^2}{A^2}\left(\lambda_1 - \frac{\nu_1\varkappa_1}{\sigma_1}\right)\frac{M}{\varrho}\frac{\gamma-1}{\gamma}\right].\right. \tag{244}$$

Since $s/A \ll 1$, we may write this approximately as

$$\exp\left[-\frac{1}{2}k^2\left(\frac{c^2}{4\pi\sigma_1} + \frac{\mu}{\varrho}\right)\right]. \tag{245}$$

This suggests that only the resistivity and the viscosity of the medium are significant for the dissipation of hydromagnetic waves. In order that these waves be realized in any physical situation, it is essential that the time required for their amplitudes to be reduced by a factor of e^{-1} of their initial amplitudes be much larger than one period of oscillation of the wave. Thus we must have

$$\frac{2}{k^2}\left(\frac{c^2}{4\pi\sigma_1} + \frac{\mu}{\varrho}\right)^{-1} > \frac{1}{kA}. \tag{246}$$

Since $k = 2\pi/\lambda$, this relation implies that the wavelength of the disturbance must be greater than the critical wavelength λ_c given by

$$\lambda_c = \frac{\pi}{A}\left(\frac{c^2}{4\pi\sigma_1} + \frac{\mu}{\varrho}\right). \tag{247}$$

IV. Oscillations of Bounded and Non-Uniform Plasmas

1. OSCILLATIONS OF A CYLINDRICAL PLASMA

We shall now investigate the oscillations of a cylindrical plasma immersed in a uniform external magnetic field which will be assumed to be along the z-axis. In the equilibrium state the plasma is at rest and the electron charge density is neutralized by the ion charge density. We shall assume that the plasma is so dilute that direct interparticle collisions are negligible and that particles interact through long range electromagnetic interactions only. For the sake of simplicity we may, therefore, neglect the plasma pressure. For small departures from the state of static equilibrium the equations of motion for the electrons and ions, assumed singly charged, are

$$m\frac{\partial \mathbf{v}_e}{\partial t} = -e\left(\mathbf{E} + \frac{1}{c}\mathbf{v}_e \times \mathbf{B}_0\right) \tag{1}$$

$$M\frac{\partial \mathbf{v}_i}{\partial t} = e\left(\mathbf{E} + \frac{1}{c}\mathbf{v}_i \times \mathbf{B}_0\right), \tag{2}$$

together with Maxwell equations

$$\nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J} \quad \text{and} \quad \nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \tag{3}$$

where

$$\mathbf{J} = eN(\mathbf{v}_i - \mathbf{v}_c). \tag{4}$$

We have neglected the displacement current as we are interested only in low frequency oscillations satisfying the condition $\omega \ll ck$. From the definitions of the mass density and the mean mass velocity, we obtain

$$\frac{1}{Ne^2}\frac{\partial \mathbf{J}}{\partial t} = \left(\frac{1}{M} + \frac{1}{m}\right)\mathbf{E} + \frac{1}{c}\left(\frac{1}{M}\mathbf{v}_\mathrm{i} + \frac{1}{m}\mathbf{v}_e\right) \times \mathbf{B}_0 \simeq \frac{1}{m}\left(\mathbf{E} + \frac{1}{c}\mathbf{v}_e \times \mathbf{B}_0\right), \tag{5}$$

and

$$\varrho\frac{\partial \mathbf{v}}{\partial t} = \frac{1}{c}\mathbf{J} \times \mathbf{B}_0, \tag{6}$$

neglecting terms of order m/M. We now need a relation connecting $\mathbf{v}_e$, $\mathbf{v}$, and $\mathbf{J}$. From the definitions of $\mathbf{J}$ and $\mathbf{v}$ we readily obtain

$$\mathbf{v}_\mathrm{e} \simeq \mathbf{v} - \frac{1}{Ne}\mathbf{J}. \tag{7}$$

From Eqs. (5) and (7) we get

$$4\pi\omega_p{}^{-2}\frac{\partial \mathbf{J}}{\partial t} = \mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{B}_0 - \frac{1}{cNe}\mathbf{J} \times \mathbf{B}_0. \tag{8}$$

$$\begin{vmatrix} \omega^2 - \alpha A^2 k^2 + i\frac{\mu}{\varrho}k^2\omega - \left(\varphi s^2 - \frac{i}{3}\frac{\mu}{\varrho}\omega\right)k^2\sin^2\theta & -\left(\varphi s^2 - \frac{i\omega}{3}\frac{\mu}{\varrho}\right)k^2\sin\theta\cos\theta \\ -\left(\varphi s^2 - \frac{i}{3}\omega\frac{\mu}{\varrho}\right)k^2\sin\theta\cos\theta & \omega^2 + \frac{i\mu}{\varrho}k^2\omega - \left(\varphi s^2 - \frac{i}{3}\frac{\mu}{\varrho}\omega\right)k^2\cos^2\theta \end{vmatrix} = 0. \tag{239}$$

As we are interested in frequencies of oscillation much below the electron cyclotron frequency, we may drop the term on the left hand side of Eq. (8) and obtain for Ohm's law:

$$\mathbf{E} + \frac{1}{c}\,\mathbf{v} \times \mathbf{B}_0 - \frac{1}{cNe}\,\mathbf{J} \times \mathbf{B}_0 = 0. \tag{9}$$

From Eqs. (3) and (9) we then obtain

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \left(\mathbf{v} \times \mathbf{B}_0 - \frac{1}{Ne}\,\mathbf{J} \times \mathbf{B}_0\right). \tag{10}$$

Equations (6) and (10) form the basis of the characteristic value problem which must be solved subject to appropriate boundary conditions. We now assume that the t, θ, and z dependence of all quantities has the form $\exp i\,(-\omega t + m\theta + kz)$. Equations (6) and (10) then lead to

$$-\frac{\omega^2}{A^2}\mathbf{B} = \nabla \times \left\{[(\nabla \times \mathbf{B}) \times \mathbf{n}] \times \mathbf{n} + i\,\frac{\omega}{\omega_i}(\nabla \times \mathbf{B}) \times \mathbf{n}\right\} \tag{11}$$

where A denotes the Alfvén velocity and $\mathbf{n}$ is a unit vector along $\mathbf{B}_0$. On carrying out the normal mode analysis, we obtain the following equations for the various components of $\mathbf{B}$:

$$B_r = -\Lambda i k\left[-\frac{m}{r}\Omega\,\frac{\omega^2}{A^2}B_z + \left(k^2 - \frac{\omega^2}{A^2} - k^2\Omega^2\right)\frac{dB_z}{dr}\right], \tag{12}$$

$$B_\theta = \Lambda\left[\frac{mk}{r}\left(k^2 - \frac{\omega^2}{A^2} - \Omega^2 k^2\right)B_z - k\,\Omega\,\frac{\omega^2}{A^2}\,\frac{dB_z}{dr}\right], \tag{13}$$

and

$$\frac{1}{r}\frac{d}{dr}r\,\frac{dB_z}{dr} + \left(\chi^2 - \frac{m^2}{r^2}\right)B_z = 0, \tag{14}$$

where

$$\frac{1}{\Lambda} = \left(k^2 - \frac{\omega^2}{A^2}\right)^2 - k^4\Omega^2, \tag{15}$$

$$\chi^2 = -\frac{1}{\Lambda}\,\frac{1}{k^2 - (\omega^2/A^2)\,k^2\Omega^2}, \tag{16}$$

and $\Omega = \omega/\omega_i$. The solution of Eq. (14) which is regular on the axis of the cylinder is given by

$$B_z = B_1 J_m(\chi r), \tag{17}$$

where B_1 is an arbitrary multiplicative constant. Having determined B_z, B_r, and B_θ are readily obtained from Eqs. (12) and (13). This determines the perturbation in the magnetic field inside the plasma.

1.1 The Perturbation in the Vacuum Field

The vacuum magnetic field satisfies the equations

$$\mathbf{B} = \nabla\psi \qquad \text{and} \qquad \nabla^2\psi = 0. \tag{18}$$

The solution of Eq. (18) which is bounded at infinity is given by

$$\psi = B_2 K_m(kr), \tag{19}$$

where B_2 is another constant, $K_m(x)$ is the Bessel function of purely imaginary argument (the notation is that of Watson), and we have suppressed the factor $\exp i\,(m\theta + kz)$.

1.2 The Boundary Conditions

The boundary conditions which must be satisfied at the perturbed boundary of the plasma are

$$\Delta\,(\mathbf{B} \cdot \mathbf{n}) = 0, \tag{20}$$

and

$$\Delta\left(\frac{\mathbf{B}^2}{8\pi}\right) = 0, \tag{21}$$

where $\Delta\,(X)$ denotes the jump experienced by the quantity X at the interface. The linearized forms of these equations are

$$B_r^{\,p} = B_r^{\,v} \qquad (r = R) \tag{22}$$

and

$$B_z^{\,p} = B_z^{\,v} \qquad (r = R), \tag{23}$$

where have now used the superscripts p and v to indicate the quantities referring to the plasma and vacuum respectively. Using these boundary conditions, and after some straightforward reductions, we obtain the dispersion relation

$$k^2\Lambda\left[m\Omega\,\frac{\omega^2}{A^2} + \left(k^2 - \frac{\omega^2}{A^2} - \Omega^2 k^2\right)\chi R\,\frac{J'_m(\chi R)}{J_m(\chi R)}\right] = \frac{kRK'_m(kR)}{K_m(kR)}, \tag{24}$$

For a specified k, A, and, Ω Eqs. (15) and (16) can be solved to give ω in terms of χ. When the ω thus obtained is substituted into Eq. (24), it admits an infinite number of discrete roots for χ for prescribed values of k, A, and Ω; denote these roots by χ_n. The dispersion relation then takes the form

$$\frac{\omega^2}{\omega_i^2} = \left(1 - \frac{\omega^2}{k^2 A^2}\right)\left(1 - \frac{\omega^2}{A^2}\,\frac{1}{k^2 + \chi_n^2}\right). \tag{25}$$

1.21 Low frequency oscillations

Consider first the case when $\omega/\omega_i \ll 1$. Then Eq. (25) yields the roots

$$\omega^2 = k^2 A^2 \tag{26}$$

and

$$\omega^2 = A^2\,(k^2 + \chi_n^2), \tag{27}$$

which correspond to the ordinary and the extra-ordinary hydromagnetic waves.

1. 22 Oscillations near the ion cyclotron frequency

We now consider the case when $\omega/kA \ll 1$. Eq. (25) then yields

$$\omega^2 = \omega_i^2\left[1 - \frac{\omega_i^2}{k^2 A^2} - \frac{\omega_i^2}{A^2\,(k^2 + \chi_n^2)}\right). \tag{28}$$

These oscillations are referred to as *ion cyclotron oscillations.*

In ion cyclotron oscillations the ion velocities are circularly polarized and the sense of rotation is the same as that for a free ion moving in a magnetic field. The electric field is elliptically polarized. One may decompose this electric field into two circularly polarized components, one of which will have the same sense of rotation as the ion motion in the wave. However, the ion velocity is 90° out of phase with that component of the electric

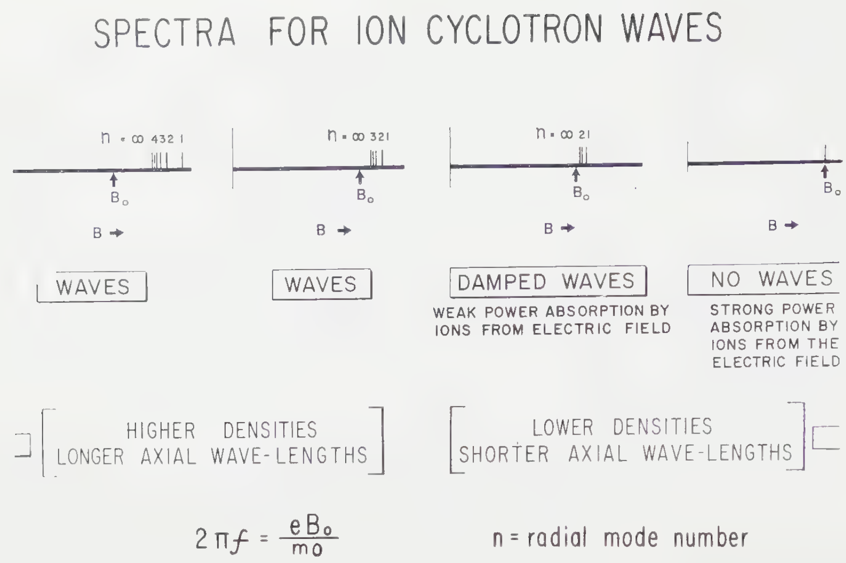
SPECTRA FOR ION CYCLOTRON WAVES
n = ∞ 4 3 2 1
n = ∞ 3 2 1
n = ∞ 2 1
B₀
B →
WAVES
WAVES
DAMPED WAVES
WEAK POWER ABSORPTION BY IONS FROM ELECTRIC FIELD
NO WAVES
STRONG POWER ABSORPTION BY IONS FROM THE ELECTRIC FIELD
HIGHER DENSITIES
LONGER AXIAL WAVE-LENGTHS
LOWER DENSITIES
SHORTER AXIAL WAVE-LENGTHS
2πf = eB₀/m₀
n = radial mode number

Fig. 14

field which rotates in the same sense as the ion. Consequently, there is no absorption of energy by the ions from the waves.

However, in the foregoing discussion we have not taken into account the thermal motion of the ions, and/or we have neglected also the collisions (short range) between the ions and electrons or the like particles. In either case damping of the ion cyclotron waves will result; in the former case we have Landau damping and in the latter case we have the usual collisional damping. When damping occurs there appears a component of the elliptically polarized electric field which is in phase with the ion motions. This leads to absorption of energy by the ions from the electric field of the wave. Consequently, this phenomenon is of fundamental interest in connection with the heating of the plasma.

In the experimental situation one has the plasma in a magnetic field and one superposes on this a transverse electric field which oscillates at the ion cyclotron frequency. As the ion motion is in phase with the electric field, energy is fed into ion motion, and the ions will move in a spiral of increasing radius.

At higher densities, say 10^{13} ions/cm^3, this single particle picture is no longer valid. Ion currents induce an electric field which has a decelerating effect on the ions, and heating a moderately dense plasma at the ion cyclotron frequency becomes rather inefficient. However, if we keep the frequency of the oscillating electric field fixed, but go to slightly stronger magnetic field, we find the existence of natural oscillations of the plasma. These natural oscillations, which we call "*ion cyclotron waves,*" are, in fact, the short-wave, low-density limit for the extra-ordinary hydromagnetic waves described by Alfvén and Aström.

Fig. 14 shows the spectra for ion cyclotron waves. The resonances are indicated as lines and are plotted against magnetic field strength. We first look at the picture denoted by "no waves." Here we have low densities and, as a consequence, single particle accelerations of the ions in the electric field. The ions move in growing spiral orbits and there is strong power absorption by the ions from the electric field. There are no waves at this very low density limit.

For moderately high densities (Fig. 14) there are ion cyclotron waves. The different vertical lines indicate the different modes of oscillation, and for the high density condition all of these modes occur at magnetic fields considerably stronger than the cyclotron field B_0. Once the ion cyclotron wave has been set up, the ions do not absorb energy from the electric fields. Therefore, there is no damping of the wave.

The second picture in Fig. 14 shows the spectrum for ion cyclotron waves for somewhat lower densities. The various modes now occur at magnetic field values which are somewhat closer to B_0. There is still no damping of the waves. In the third picture the density is quite low and the modes now occur at magnetic fields very close to B_0. This is an intermediate case between the ion cyclotron wave resonance and the single particle resonance. For this condition we have waves, but the ions begin to move in growing spiral orbits and pick up energy from the electric field. In this case we have ion cyclotron waves which are slightly damped.

The transition from the undamped waves to the slightly damped waves is particularly interesting because it provides a method for thermalizing ion cyclotron waves, that is, for transforming the wave energy quickly into heat. It turns out that it is equally valid to talk of wavelengths instead of densities, and it is the short wavelengths which are damped. To illustrate a plasma heating scheme, we make an analogy to ocean waves running up on a beach. A cross section of an ocean beach is shown in Fig. 15. The waves are moving in toward the shore and, because the water is getting more and more shallow, the wave length of the ocean waves

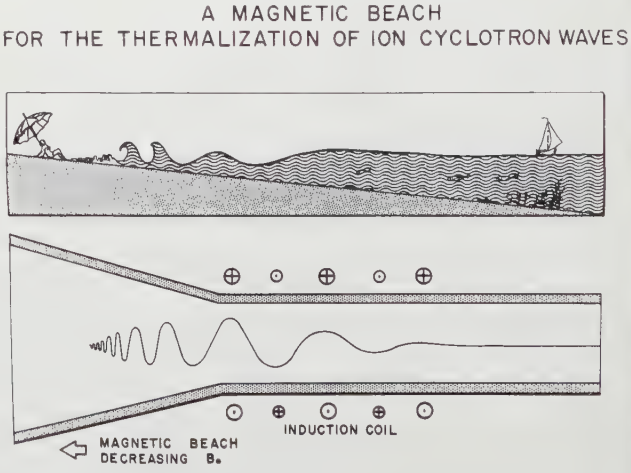

Fig. 15

becomes shorter and shorter. Finally, in the shallow water the waves are unable to propagate and they "break;" the wave energy is transformed into heat. Similarly, one might use an induction coil to generate rather long wavelength ion cyclotron waves and allow these waves to propagate along the magnetic lines of force through a region where the magnetic field tapers off and becomes somewhat weaker. The wavelength becomes shorter and shorter and, finally, when the wavelength is sufficiently short the waves damp out and the wave energy goes into heating the ions. It woud be in this "magnetic beach" region that the real heating of the gas would take place.

2. Radiation by Plasma Oscillation

We now wish to exhibit the possibility of radiation by plasma oscillations. Firstly, it may be remarked here that this problem is mathematically quite analogous to the scattering of electromagnetic radiation by a dielectric. Consider, then, a simple equilibrium configuration in which the ions are held fixed with a uniform density N and the electrons, whose thermal motions will be neglected, are distributed likewise inside a sphere of radius R. From the linearized equation of motion

$$m \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} = -e\mathbf{E} \tag{29}$$

and the linearized Maxwell equations, we readily obtain

$$c^2 \nabla \times \nabla \times \mathbf{E} = (\omega^2 - \omega_p^2)\,\mathbf{E}, \tag{30}$$

where we have assumed the time dependence to be of the form exp $(-i\omega t)$, and ω_p denotes the electron plasma frequency. From Eq. (30) it follows that

$$\nabla \cdot [(\omega^2 - \omega_p^2)\,\mathbf{E}] = 0. \tag{30a}$$

Inside the sphere ω_p is constant; consequently, if $\omega^2 \ddagger \omega_p^2$ Eq. (30a) leads to

$$\nabla \cdot \mathbf{E} = 0 \tag{31}$$

for both inside and outside the plasma.

Because of the symmetry of the problem, it is convenient to seek a solution of Eq. (30) in terms of vector spherical harmonics:

$$\mathbf{A} = \hat{e}_r\, Y_l^m(\theta, \varphi,), \tag{32}$$

$$\mathbf{B} = \frac{r}{[l(l+1)]^{1/2}} \nabla Y_l^m(\theta, \varphi), \tag{33}$$

and

$$\mathbf{C} = \frac{1}{[l(l+1)]^{1/2}} \nabla \times [\mathbf{r}\, Y_l^m(\theta, \varphi)], \tag{34}$$

where $Y^m{}_l$ is the normalized spherical harmonic. **A**, **B**, and **C** constitute a complete set of basic orthogonal vectors on the surface of a sphere; namely,

$$\iint \mathbf{P}(l, m; \theta, \varphi) \cdot \mathbf{Q}(l, m; \theta, \varphi)\, d\Omega = 0, \tag{35}$$

where **P** and **Q** are any two distinct vectors **A**, **B**, and **C**. The normalization condition is

$$\iint \mathbf{P}(l_1, m_1; \theta, \varphi) \cdot \mathbf{P}(l_2\, m_2; \theta, \varphi)\, d\Omega = \delta_{l_1 l_2} \delta_{m_1 m_2}. \tag{36}$$

The spherical harmonics satisfy the equation

$$\nabla^2 Y_l^m(\theta, \varphi) = -\frac{1}{r^2} l(l+1)\, Y_l^m(\theta, \varphi). \tag{37}$$

We now write

$$\mathbf{E} = \frac{F}{r^2}\mathbf{A} + \frac{G}{r}\mathbf{B} + \frac{H}{r}\mathbf{C}, \tag{38}$$

where F, G, and H are functions of r alone. On substituting Eq. (38) into Eq. (31) we obtain

$$\frac{dF}{dr} = \left[l(l+1)\right]^{1/2} G. \tag{39}$$

On substituting Eq. (38) into Eq. (30) and making use of the relation (35), we obtain the two independent equations

$$\frac{d^2H}{dr^2} + \left[\frac{1}{c^2}(\omega^2 - \omega_p^2) - \frac{1}{r^2} l(l+1)\right] H = 0 \tag{40}$$

and

$$\frac{dG}{dr} + \frac{1}{[l(l+1)]^{1/2}} \left[\frac{1}{c^2}(\omega^2 - \omega_p^2) - \frac{1}{r^2} l(l+1)\right] F = 0. \tag{41}$$

The problem, then, is to solve Eqs. (39)—(41) subject to appropriate boundary conditions.

2.1 The boundary conditions

From Eq. (30a) it follows upon integration across the interface that we must have

$$(\omega^2 - \omega_p^2)\, \mathbf{E}_1 \cdot \mathbf{n} = \omega^2\, \mathbf{E}_2 \cdot \mathbf{n}, \tag{42}$$

where the subscripts 1 and 2 refer to the plasma and vacuum sides respectively and **n** denotes the unit outward normal to the surface. Since only **A** has a radial component, it follows from Eq. (42) that

$$(\omega^2 - \omega_p^2)\, F_1 = \omega^2 F_2 \quad \text{at } r = R. \tag{43}$$

It follows from $c\nabla \times \mathbf{E} = -\partial\mathbf{B}/\partial t$ that the tangential components of **E** are continuous at the interface. This requires that

$$G_1 = G_2 \quad \text{at } r = R, \tag{44}$$

and

$$H_1 = H_2 \quad \text{at } r = R. \tag{45}$$

Integrating Eq. (40) across the interface, we get

$$\frac{dH_1}{dr} = \frac{dH_2}{dr} \quad \text{at } r = R. \tag{46}$$

The conditions (45) and (46) may be combined to read

$$\frac{d}{dr} \ln H_1 = \frac{d}{dr} \ln H_2. \tag{47}$$

2.2 Solution of the problem

The solution of Eq. (40), which is regular at the center of the sphere, is given by

$$H = H_i r^{(1/2)} J_{l+1/2}(k_1 r), \tag{48}$$

where

$$k_1^2 = \frac{1}{c^2}(\omega^2 - \omega_p^2) \tag{49}$$

and H_i is an arbitrary multiplicative constant. It can be seen readily from Eqs. (39) and (41) that F satisfies the same equation as H; consequently, a similar solution holds for F. G is then determined by Eq. (39).

2.3 The case of a perfectly conducting boundary

An interesting situation which may approximate cavity resonators is the one where the plasma is enclosed by perfectly reflecting walls. The boundary conditions to be imposed in this case are that E_θ and E_φ vanish at $r = R$. This requires that

$$J_{l+1/2}(k_1 R) = 0. \tag{50}$$

Let the roots of Eq. (50) be denoted by $j_{l,m}$. From Eqs. (49) and (50) we than obtain the dispersion relation

$$\omega^2 = \omega_p^2 + \frac{c^2}{R^2} j^2_{l,m}. \tag{51}$$

It may be remarked here that as $R \to \infty$, $\omega \to \omega_p$, the natural plasma frequency.

2.4 The case of free boundary

In the case of a free boundary, the inside solution must be matched to the one outside with due regard to the boundary conditions. The equations governing the vacuum field can be obtained readily from Eqs. (40) and (41) by setting $\omega_p^2 = 0$. Thus H satisfies the equation

$$\frac{d^2 H}{dr^2} + \left[k_2^2 - \frac{l(l+1)}{r^2}\right] H = 0, \tag{52}$$

where $k_2 = \omega/c$. The solution of Eq. (52) is given by

$$H = r^{1/2}\left[H_0 H^{(1)}_{l+1/2}(k_2 r) + H_{01} H^{(2)}_{l+1/2}(k_2 r)\right], \tag{53}$$

where $H_\nu^{(1)}$ and $H_\nu^{(2)}$ are the two kinds of Hankel functions whose asymptotic forms are given by ($k_2 r \gg 1$)

$$H_\nu^{(1)}(z) \sim \left(\frac{2}{\pi z}\right)^{1/2} e^{i[z-(\pi/4)(2\nu+1)]} \tag{54}$$

and

$$H_\nu^{(2)}(z) \sim \left(\frac{2}{\pi z}\right)^{1/2} e^{-i[z-(\pi/4)(2\nu+1)]}. \tag{55}$$

In Eq. (53) H_0 and H_{01} are constants of integration. We have assumed the time dependence of the wave to be of the form $\exp - i\omega t$. Therefore, in order that at large distances from the sphere we have only an outgoing wave, we must set $H_{01} = 0$. Thus the permissible solution is

$$H = H_0 r^{1/2} H^{(1)}_{l+1/2}(k_2 r). \tag{56}$$

Applying the boundary condition (47), we then obtain

$$\frac{d}{dr} \ln J_{l+1/2}(k_1 r) = \frac{d}{dr} \ln H^{(1)}_{l+1/2}(k_2 r). \tag{57}$$

We now restrict ourselves to the special case where $k_1 R \gg 1$ and $k_2 R \gg 1$. We may then use the asymptotic expansion

$$J_\nu(z) \sim \left(\frac{2}{\pi z}\right)^{1/2} \cos\left[z - \frac{\pi}{4}(2\nu+1)\right] \tag{58}$$

and Eq. (54) for $H_\nu^{(1)}$. Eq. (57) then leads to

$$-k_1 \tan\left[k_1 R - \frac{\pi}{2}(l+1)\right] = i k_2. \tag{59}$$

Using the relation

$$\text{arc tan } z = \frac{1}{2i} \ln \frac{1+iz}{1-iz}, \tag{60}$$

we obtain from Eq. (59)

$$\frac{R}{c}(\omega^2 - \omega_p^2)^{1/2} = \frac{\pi}{2}(2n+l+2) + \frac{1}{2i} \ln \frac{\omega + (\omega^2-\omega_p^2)^{1/2}}{\omega - (\omega^2-\omega_p^2)^{1/2}}. \tag{61}$$

The notion of a radiative solution implies that, at a fixed point in space, the amplitude of the perturbation should die out asymptotically in time on account of a finite amount of energy available for radiation in any admissible initial perturbation of the plasma. This implies that ω has a negative imaginary part. We consider oscillations close to the plasma frequency and set

$$\omega = \omega_p + \delta - i\gamma, \tag{62}$$

where γ is positive, and δ and γ are both assumed small. On substituting Eq. (62) into Eq. (61) and retaining terms to the lowest significant order, we obtain

$$\frac{\delta}{\gamma} \doteq \frac{c}{2\omega_p R}\left(\omega_p^2 \frac{R^2}{c^2} - 1\right) \tag{63}$$

and

$$2\gamma\left[2\frac{R}{c} + \frac{c}{2\omega_p^2 R}\left(\omega_p^2 \frac{R^2}{c^2} - 1\right)^2\right] = \left[\pi\left(n+1+\frac{l}{2}\right)\right]^2. \tag{64}$$

We now consider the physically interesting situation where the time taken by the wave to traverse the radius of the sphere is much larger than the period of a plasma oscillation. That is,

$$\frac{R}{c}\omega_p = \frac{\tau_t}{\tau_p} \gg 1, \tag{65}$$

where

$$\tau_t = \frac{R}{c} \text{ and } \tau_p = \frac{1}{\omega_p}. \tag{66}$$

From Eqs. (63) and (64) we obtain to the lowest significant order

$$\gamma = \omega_p \left(\frac{\tau_p}{\tau_t}\right)^3 \left[\pi\left(n+1+\frac{l}{2}\right)\right]^2 \tag{67}$$

and

$$\sigma = \frac{1}{2}\omega_p \left(\frac{\tau_p}{\tau_t}\right)^2 \left[\pi\left(n+1+\frac{l}{2}\right)\right]^2. \tag{68}$$

This establishes the possibility of radiation by plasma oscillations. However, it is not possible to determine the intensity of the radiation emitted within the framework of the linearized theory presented here and in the absence of any knowledge of the excitation mechanism for the oscillations.

In the preceding calculations it was tacitly assumed that $\omega^2 \neq \omega_p^2$. There are, however, situations for which

$\omega^2 = \omega_p^2$. It will be recalled from § 2 of Section III that within any region of a cold plasma of uniform density one can construct arbitrary wave packets of longitudinal waves characterized by

$$\nabla \times \mathbf{E} = 0 \text{ and } \nabla \cdot \mathbf{E} \neq 0\,, \tag{69}$$

which are non-dispersive and which do not couple to the radiation field characterized by

$$\nabla \cdot \mathbf{E} = 0\,. \tag{70}$$

The longitudinal oscillations, of course, can exist also in the interior of the sphere considered here, but clearly they do not radiate.

Those radiating modes involving the functions F and G, discussed previously, are associated, however, with a fluctuating surface density and do radiate.

The sharp transition between plasma and vacuum assumed here is merely a convenient mathematical device for representing a rapid, but continuous, transition in density. Clearly the radiation associated with the fluctuating surface charge can be viewed as arising from a coupling between longitudinal and transverse fields induced by the density gradient. In fact, one can verify readily that for the case where the density is a function of position, the decomposition into solenoidal and irrotational waves is no longer possible. For the general case, however, the calculations become much more difficult and the radiation associated with charge separation has, as yet, been explored only incompletely.

3. OSCILLATIONS IN NON-UNIFORM PLASMAS

We now consider the longitudinal electron oscillations in non-uniform plasmas in which temperature and/or density gradients are maintained by some agency. The model we take for the steady state is one in which the electron velocity distribution is assumend to be isotropic against a fixed background of positive ions, resulting in an electric field which balances the pressure forces in the equilibrium state. The force balance condition is, then,

$$\nabla p = -Ne\mathbf{E}\,, \tag{71}$$

where $p\ (= N\Theta)$ and $\mathbf{E}$ denote the pressure and the electric field in equilibrium respectively. It is also assumed that there is no magnetic field. If we assume Θ to be constant, Eq. (71) can be integrated to give

$$N = N_0 \exp\frac{e\varphi}{\Theta}\,, \tag{72}$$

where φ is the electrostatic potential given by $\mathbf{E} = -\nabla\varphi$ and N_0 is a constant. Eq. (72) is the well known density law for an isothermal atmosphere. On the other hand, if we assume N to be constant, Eq. (71) gives the energy integral

$$\mathfrak{E} = \Theta - e\varphi\,. \tag{73}$$

In general, it follows from Eq. (71) that

$$p = f(N)\,, \tag{74}$$

where f is an arbitrary function of its argument.

The equations governing the small departures from the state of equilibrium are (cf. Eqs. (II-47), (II-48), and (II-119))

$$\frac{\partial n}{\partial t} + \nabla \cdot (N\mathbf{v}) = 0\,, \tag{75}$$

$$mN\frac{\partial \mathbf{v}}{\partial t} = -\nabla \cdot \mathsf{p}_1 - ne\mathbf{E} - Ne\mathbf{E}_1\,, \tag{76}$$

and

$$\frac{\partial}{\partial t}\mathsf{p}_1 + \nabla \cdot (\mathbf{v}p)\,\mathbf{I} + p\nabla\mathbf{v} + p(\nabla\mathbf{v})^T = 0\,, \tag{77}$$

where n denotes the perturbation in the electron density and p_1, the perturbation in the stress tensor. The perturbation in the electric field $\mathbf{E}_1$ for the longitudinal oscillations under consideration is given by the Poisson equation

$$\nabla \cdot \mathbf{E}_1 = -4\pi e n\,. \tag{78}$$

Introducing $\boldsymbol{\xi}$, the Lagrangian displacement of an element of the fluid, Eqs. (75), (77), and (78) can be integrated to give

$$n = -\nabla \cdot (N\boldsymbol{\xi})\,, \tag{79}$$

$$\mathsf{p}_1 = -\nabla \cdot (\boldsymbol{\xi}p)\,\mathbf{I} - p\,(\nabla\boldsymbol{\xi}) - p\,(\nabla\boldsymbol{\xi})^T\,, \tag{80}$$

and

$$\mathbf{E}_1 = 4\pi e N\boldsymbol{\xi}. \tag{81}$$

Using the foregoing results in Eq. (76) we obtain

$$mN\frac{\partial^2\boldsymbol{\xi}}{\partial t^2} = p\nabla\left[\frac{1}{p}\,\nabla\cdot(p\boldsymbol{\xi})\right] + \frac{1}{p}\,\nabla p\,(\boldsymbol{\xi}\cdot\nabla p)$$
$$-\frac{1}{N}f'\nabla n\,(\boldsymbol{\xi}\cdot\nabla N) + \nabla.\left[p\nabla\boldsymbol{\xi} + p\,(\nabla\boldsymbol{\xi})^T\right] - mN\omega_p^2\boldsymbol{\xi}. \tag{82}$$

We now multiply Eq. (82) by $\boldsymbol{\xi}^*$ and integrate over the volume of the plasma. Assuming that $\boldsymbol{\xi}$ vanishes at the boundary, Eq. (82) leads to

$$\omega^2\int mN|\boldsymbol{\xi}|^2\,d\tau = \int\Biggl\{\left[\frac{1}{p^{1/2}}\,|\nabla\cdot(p\boldsymbol{\xi})|\right]^2 - \left[\frac{1}{p^{1/2}}|\boldsymbol{\xi}\cdot\nabla p|\right]^2$$
$$+\left[\left(\frac{f'}{N}\right)^{1/2}|\boldsymbol{\xi}\cdot\nabla N|\right]^2 + 2p\,(\nabla\boldsymbol{\xi})_{\text{sym}} : \nabla\boldsymbol{\xi}^* + mN\omega_p^2|\boldsymbol{\xi}|^2\Biggr\}d\tau. \tag{83}$$

Now all the terms except the second on the right hand side of Eq. (83) are positive definite. Thus the system will be unstable only if the second term is the dominant one on the right hand side of Eq. (83).

3.1 Density gradients

Let us first specialize to the case where there exist density gradients in a plasma at uniform temperature. Then $f = N\Theta$, where Θ is a constant and the second and the third terms on the right hand side of Eq. (83) cancel each other. Consequently, ω^2 is always positive definite and the system is *stable*.

3.2 Temperature gradients

We now consider the case when $\Theta = \Theta\,(\mathbf{x})$ and N is constant. Then the third term on the right hand side of Eq. (83) vanishes. For instability to occur the second term must be capable of having a large negative value compared to the fifth term. In order of magnitude, if b denotes the characteristic scale of variation of Θ, then for instability we must have

$$mN\omega_p^2 < \frac{N\Theta}{b^2}. \tag{84}$$

Eq. (84) can be written in the form

$$b^2 < \frac{\Theta}{m\omega_p^2} = \lambda_D^2. \tag{85}$$

Thus, for instability to occur we must have significant temperature gradients over lengths smaller than the Debye length. However, at such small lengths the macroscopic theory is inapplicable.

We may thus conclude that electron oscillations in non-uniform plasmas with arbitrary density or temperature gradients are macroscopically stable.

V. Non-Linear Oscillations

We now consider large amplitude electron oscillations in a uniform plasma and, for the sake of simplicity, restrict ourselves to one-dimensional oscillations only. Furthermore, we neglect the thermal motions of the electrons. Let x_0 denote the equilibrium position of an electron and $X(x_0)$ denote its displacement (not necessarily small) from the equilibrium position. The displaced position of the electron is then given by

$$x = x_0 + X(x_0). \tag{1}$$

In moving the distance $X(x_0)$ the electrons in the x_0 plane pass over an amount of positive charge eNX per unit area, where N denotes the equilibrium density of the electrons. If the ordering of the electrons in the x-direction is not changed, then all electrons which were originally on the positive side of a given electron remain on the positive side, while all those which were on the negative side remain on the negative side. We shall assume that the system is of infinite extent and that no charge enters or leaves the system at infinity. The equation of motion of an electron is given by

$$m\frac{d^2X}{dt^2} = -eE, \tag{2}$$

where, by Gauss's theorem, the electric field is given by

$$E = 4\pi NeX. \tag{3}$$

From Eqs. (2) and (3) we obtain for the equation of motion

$$\frac{d^2X}{dt^2} = -\omega_p^2 X. \tag{4}$$

where $\omega_p^2 = 4\pi Ne^2/m$. The general solution of Eq. (4) is given by

$$X = X_1(x_0)\sin\omega_p \mathrm{t} + X_2(x_0)\cos\omega_p t, \tag{5}$$

where $X_1(x_0)$ and $X_2(x_0)$ are some arbitrary functions of x_0. Thus each electron executes simple harmonic motion about its equilibrium position, independent of its amplitude and the motion of the neighboring electrons so long as the ordering of the electrons is maintained.

The condition that the ordering is maintained can be readily derived in the following manner. Let $x_0 + X$ denote the displaced position of an electron originally at x_0, while $x_0 + \Delta x_0 + X + \Delta X_0$ denotes the displaced position of an electron originally at $x_0 + \Delta x_0$. In order that the ordering be maintained, it is clear that $\Delta x_0 + \Delta X > 0$; we thus obtain

$$\frac{\partial X}{\partial x_0} > -1 \tag{6}$$

as the required condition.

An interesting special solution to Eq. (4) arises when

$$X_1 = 0 \text{ and } X_2 = A\sin kx_0, \tag{7}$$

where A is a constant and k denotes the wave number of the disturbance in the x-direction. The solution of Eq. (4) is then given by

$$x = x_0 + A\sin kx_0\cos\omega_p t \tag{8}$$

and

$$E = 4\pi NeA\sin kx_0\cos\omega_p t. \tag{9}$$

At $t = 0$ Eqs. (8) and (9) reduce to

$$x = x_0 + A\sin kx_0 \tag{10}$$

and

$$E = 4\pi NeA\sin kx_0. \tag{11}$$

We may solve Eq. (10) for x_0 as a function of x (this is most simply done numerically) and thus obtain E as a function of x. The ratio $E/E_{\max}$ is plotted in Fig. 16 as a function of x for various values of A. For small values of A the curves are essentially sine waves. For large values of A one gets a multiple valued curve which is, of course, meaningless. What has happened is that the ordering of the electrons has been destroyed. The crest of the wave falls into the trough and we have a wave breaking process somewhat similar to the breaking of water-waves, and we may expect the waves to be dissipated by this breaking mechanism converting the wave energy into the thermal energy of the electrons.

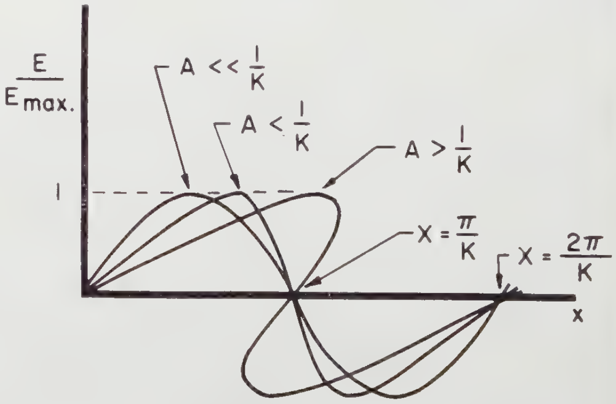

Fig. 16 The behavior of $E/E_{\max.}$ as a function of x for various values of A at time $t = 0$.

A convenient mechanical model which exhibits the breaking process is a series of elastic pendulums constrained to move along a straight line. For small displacements the pendulums swing independently of each other. However, if the amplitude of the displacement becomes large, the bobs of the pendulums strike one another, destroying their periodic motion.

The breaking of large amplitude waves has been investigated by Dawson for the one dimensional case under various initial conditions. Firstly, consider the case when the initial displacement vanishes and the initial velocity is given by

$$V = V_0 \sin k x_0. \qquad (12)$$

From Eqs. (6), (8), and (12) it follows that the critical breaking amplitude is

$$V_c = \omega_p / k. \qquad (13)$$

In his numerical calculations Dawson investigated the case when the amplitude of the wave was taken to be 7% larger than the critical amplitude given by Eq. (13).

Secondly, we consider the case when, in addition to the initial velocity given by Eq. (12), there is superimposed on the particles a certain amount of random motion. The random motion was taken to be

$$V_{th} = V_{ot} \exp\left(-V^2/V_0^2\right) \qquad (14)$$

whre V_{ot} is a constant which is proportional to the thermal energy. The effect of thermal motions on the breaking of waves is exhibited in Fig. 17 for the case where the thermal energy is 4% of the wave energy The lower curve in Fig. 17 shows the wave form for the wave at a time two plasma oscillations later. It is clear from this curve that the amplitude of the wave has reduced considerably and that the wave energy is being converted into the thermal energy of the electrons.

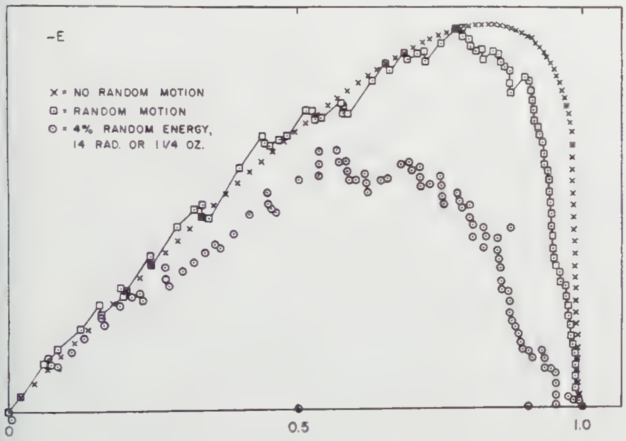

Fig. 17 The behavior of E as a function of x when the amplitude of the wave is 7% larger than the critical amplitude at which the wave breaks, for different initial conditions. The crosses show the wave form for a cold wave just as it is starting to break. The squares show the wave form for a wave with initial random energy equal to 4% of the initial wave energy, also at the start of breaking. The circles show the wave form for a warm wave two plasma periods later.

Fig. 18 shows the velocities of 270 pendulums after three oscillations for an initially cold wave. It is apparent that most of the particle velocities lie on a sine wave. However, there exist a number of particles with very high velocities, both positive and negative. These are particles which managed to get in just the right phase with the wave so as to ride it, and they account for the major loss in the energy of the wave.

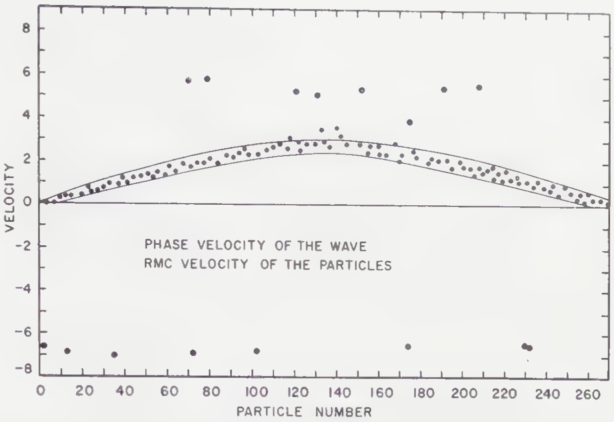

Fig. 18 The velocities of the particles after three oscillations.

In order to get a quantitative idea of the energy loss of the wave, we Fourier-analyze the displacement and the velocity in terms of the equilibrium position, i.e.,

$$X(x_i) = \sum_{n=1}^{N} A_n \sin \frac{n\pi x_i}{L} \qquad (15)$$

and

$$V(x_i) = \sum_{n=1}^{N} B_n \sin \frac{n\pi x_i}{L}, \qquad (16)$$

where L denotes the linear dimension of the system. The Fourier co-efficients A_n and B_n are given by

$$A_n = \frac{2}{N+1} \sum_{n=1}^{N} X(x_i) \sin \frac{n\pi x_i}{L}, \qquad (17)$$

and

$$B_n = \frac{2}{N+1} \sum_{n=1}^{N} V(x_i) \sin \frac{n\pi x_i}{L}. \qquad (18)$$

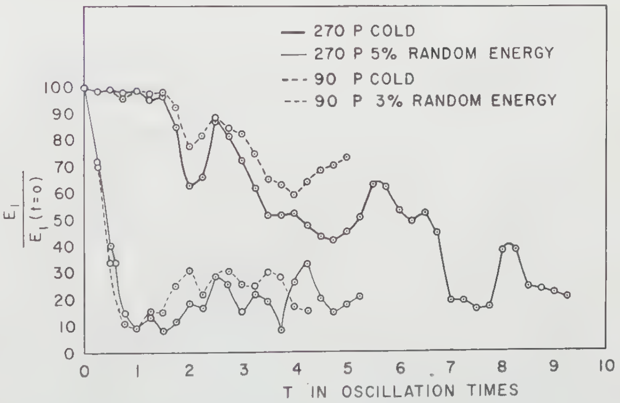

Fig. 19 The ratio of the energy in the first harmonic to its initial value for a number of different cases.

For the Fourier analysis it is convenient to use the equilibrium position rather than the actual physical position since for such a choice there is no coupling between the different modes without breaking. The results of the Fourier analysis are shown in Figs. 19 and 20. Fig. 19 shows the ratio of the energy in the first harmonic to its initial value (for a number of different cases) as a function of time. The cold wave loses very little energy during the first oscillation and a half, after which there is an almost continuous decrease in energy. As is clear from the diagram, the presence of a small amount of random motion causes the wave energy to drop to about 10% of its initial value over a period of one oscillation.

The absolute values of the amplitudes of the various Fourier modes for the case of forty-five pendulums without any random motion is shown in Fig. 20 after 2.25 oscillations. As is clear from the figure, there is no tendency to feed energy from the fundamental into any one of the higher modes. The curve is flat (with only small fluctuations superimposed on it) all the way out; only the fundamental has appreciably more energy than the rest of the modes. The energy of the wave seems to go into individual particles rather than into Fourier modes.

One may expect the system to exhibit a Poincaré cycle and return to its initial state after a sufficiently long interval of time. Simple probabilistic arguments indicate that this interval of time will indeed be very long even for a small number of charge sheets. For example, such arguments indicate that a system of five sheets should spend approximately 1/400 of the time in a state for which the energy of the fundamental is more than 95% of the total energy. A system of nine sheets would spend only one part in 1.6×10^5 of its time in such a state.

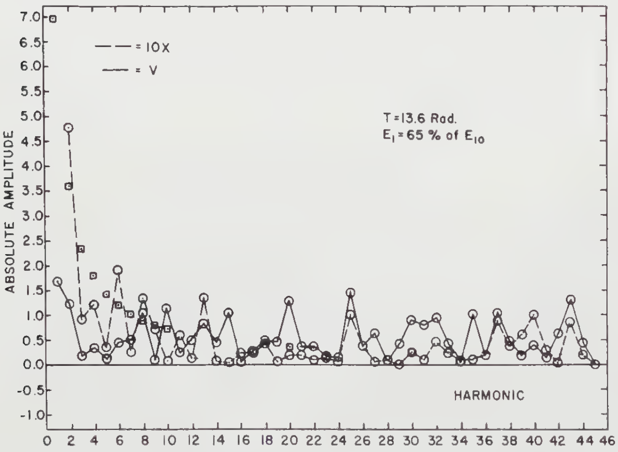

Fig. 20 The absolute values of the amplitudes for various Fourier modes for the case of forty-five particles per half wave length after 2.25 oscillations.

The recurrence time was investigated by Dawson following systems of five sheets and nine sheets for about sixty oscillations. The systems were started in the fundamental mode with amplitudes larger than the breaking amplitude. It was found that on the average there were five crossings per sheet for the five-sheet case, and ten crossings per sheet for the nine-sheet case, in one-half an oscillation time. The state of the system was sampled for 120 values of the time. It was found that the five-sheet system returned to a state in which the fundamental had 95% of the energy once during this time, while the nine-sheet system returned to a state in which the fundamental had half the energy once during this time. Fig. 21 shows the sum of 120 time-averaged velocity distribution functions for the nine-particle system. The smooth curve in Fig. 21 shows what would be expected if the motion was roughly ergodic. These results seem compatible with the probabilistic estimates.

The foregoing discussion indicates that the breaking of waves can be an effective mechanism for the dissipation of wave energy into the thermal energy of electrons and that the presence of a small amount of random motion can greatly enhance this mechanism.

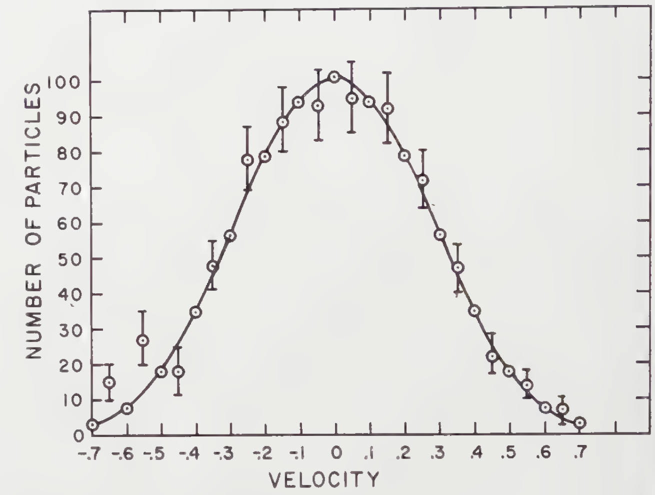

Fig. 21 The velocity spectrum of the particles.

Acknowledgements

It is a pleasure to express our indebtedness to Professor Lyman Spitzer, Jr. for his reading of the manuscript, and criticism. We are grateful to: Dr. C. R. Oberman for communicating to us his results on the low temperature approximation; to Dr. C. R. Oberman and Dr. J. Dawson for communicating to us their results on the radiation by plasma oscillations in a uniform sphere, prior to publication; to Dr. R. M. Kulsrud for reading Section I and the first part of Section II; to Dr. T. Stix for reading the first part of Section IV; and to Dr. M. Weenink for checking some of the calculations. We would also like to express our gratitude to Mrs. H. Selberg for her excellent execution of the numerical computations which were needed for the graphs. Finally, our thanks are due also to Mrs. Isabelle Murphy for capably carrying out the arduous task of typing the manuscript.

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I. INTRODUCTION

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Rayleigh's formulation of the problem was in the following terms:

"... The cloud of electrons may then be assimilated to a fluid whose properties, however, must differ in many respects from those with which we are most familiar. We suppose that the whole quantities of positive and negative are equal. The difference between them is that the positive are constrained to remain undisplaced while the negative are free to move. In equilibrium, the negative distributes itself with uniformity throughout the sphere occupied by the positive, so that the total density is everywhere zero. There is then no force at any point; but if the negative be displaced, a force is usually called into existence ..."

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I. THE BASIC EQUATIONS

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(Manuscript received on 6 June 1960)

NUCLEAR FUSION: 1, 42-46 (1960)

RAYLEIGH-TAYLOR INSTABILITIES OF A MAGNETICALLY ACCELERATED PLASMA

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High speed photography has been used to study the compression of a deuterium plasma by an axial magnetic field generated by a wide single-turn coil. In this experiment the inductance of the coil is small compared with that of the external condenser bank so that the plasma implodes at a constant acceleration of 5×10^{12} cms/sec^2. Flutes which develop on the outer surface of the plasma are interpreted as magnetohydrodynamic Rayleigh-Taylor instabilities produced by the inward acceleration of the interface separating magnetic field and plasma. The observed growth rates are in agreement with simple theory. It is shown that the ratio of the diffusivities of magnetic field and momentum determines whether the plasma resistivity or its viscosity is the dominant mechanism for damping the short wavelength instabilities. At the temperature and densities of this experiment the resistivity is primarily responsible for the damping and the wavelength of "maximum instability" calculated on this basis agrees with the observed value.

Introduction

The rapid compression of relatively dense plasmas by means of axial magnetic fields has recently received considerable attention [1]—[9] principally because this is a possible method of heating a plasma to thermonuclear temperatures. This paper describes a high speed photographic study of such a compression process in which the plasma is pinched to the centre of the discharge tube by a magnetic field generated by a wide single-turn coil. A stationary plasma confined by a uniform axial magnetic field is expected to be neutrally stable against hydromagnetic disturbances [10] but in the absence of crossed magnetic fields there is no restoring force to prevent the development of interchange instabilities of the flute type [11]. In the present experiments the plasma collapses towards the centre of the tube with approximately constant acceleration and flutes which subsequently develop on the outer surface of the plasma are interpreted as Rayleigh-Taylor instabilities produced by the inward acceleration of the interface between the magnetic field and the plasma. These instabilities are analysed on the basis of Taylors' theory and the damping by resistivity and viscosity is discussed. It is shown that the observed growth time and wavelength are in accord with simple theory.

Experimental Arrangement

A diagram of the experimental system is shown in fig. 1. A copper coil 21 cms wide surrounds a quartz tube 8.0 cms internal diameter and is connected by a single spark gap switch to a bank of capacitors supplying 3.5 kilojoules at 24 kV. The total circuit in-

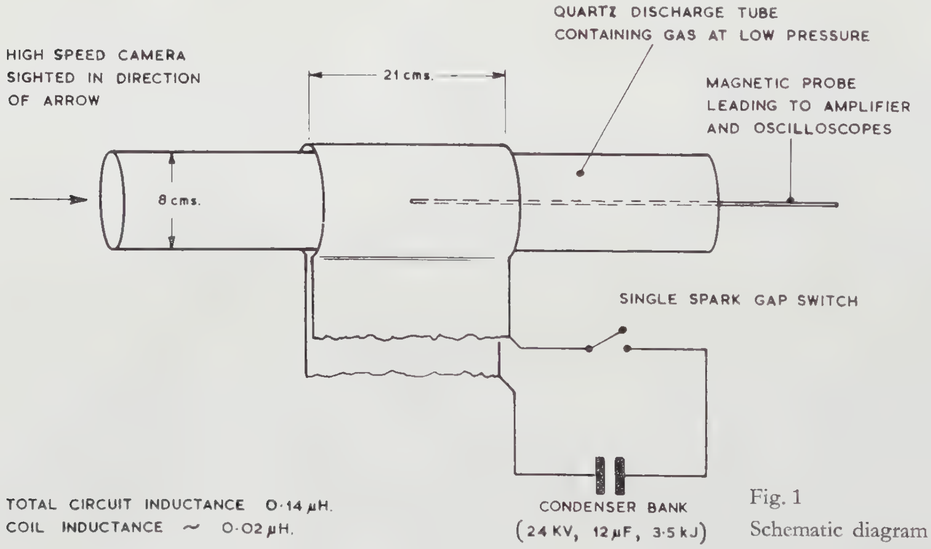
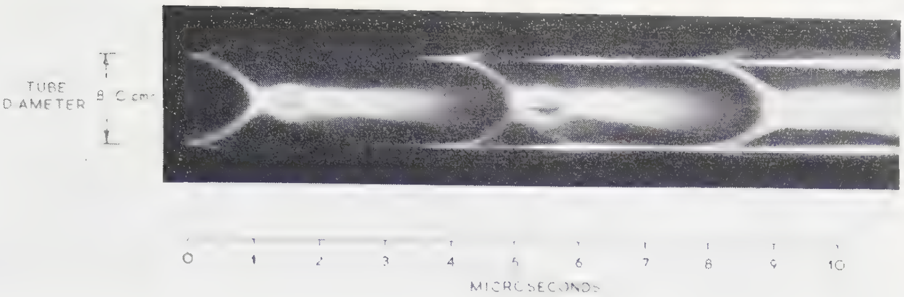


Fig. 1

Schematic diagram of the experimental arrangement.



ig. 2
treak photograph of a discharge n deuterium at an initial pressure f 80 microns. Peak magnetic ield is 13 kilogauss.

luctance is 0.14 μH of which the coil contributes about .02 μH; thus about one-seventh of the voltage appears nitially across the coil and since the inductance of the xternal circuit is so much greater than the coil inductnce, variations of the latter during the discharge have ittle effect on the voltage and current waveform. A nagnetic probe at the centre of the coil measures the ield on the axis of the tube.

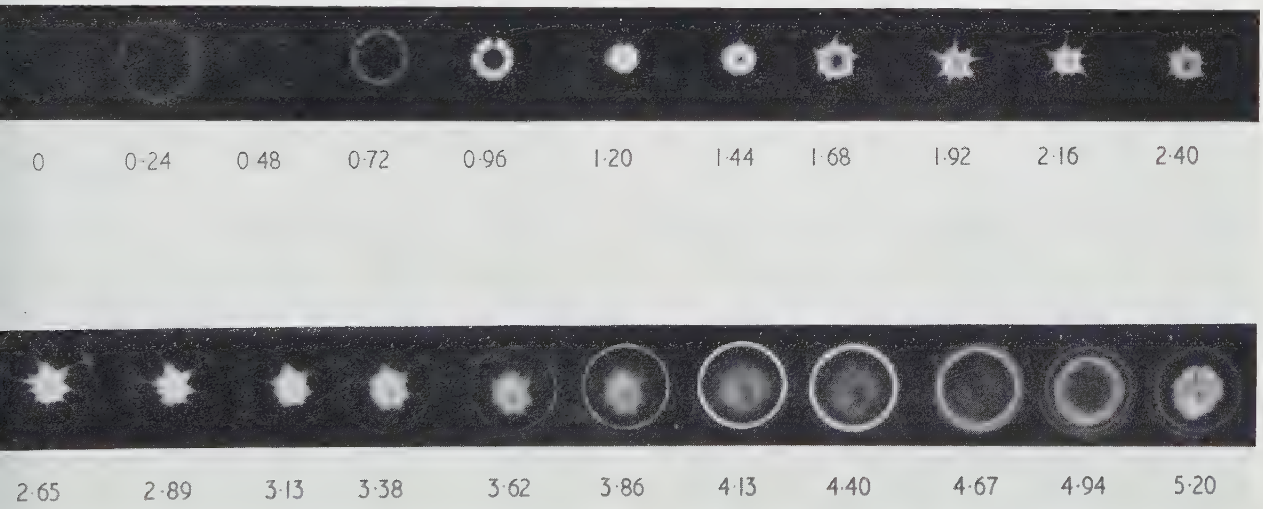
Discharges in deuterium at initial pressures in the ange 50—200 microns have been photographed axially s shown in fig. 1 using two high speed rotating mirror ameras. The streak camera records motion across a liameter and has a writing speed of up to 20 mms per nicrosecond. The other camera is fitted with 30 separate enses and gives a series of photographs of the whole ube cross-section at a framing rate of 8 $\times$ 10^6 pictures er second. The images are stabilised by using a Miller ptical system and are 8 mms in diameter.

Experimental Results

Fig. 2 shows a streak photograph of a discharge in leuterium at an initial pressure of 80 microns. The first alf-cycle in which the gas is ionised is not shown on he photograph; the gas current starts to grow at the eginning of the second half-cycle of the external curent and this time is labelled $t = 0$. The outer edge of he imploding luminous sheath is interpreted as the boundary between magnetic field and plasma and has a constant inward acceleration of 5 $\times$ 10^{12} cms/sec^2 lasting for about a microsecond. The inner boundary of the sheath is taken to be a shock wave which reaches a terminal velocity of 5.5 $\times$ 10^6 cms/sec. After the shock has reached the centre of the tube the plasma column is further compressed by the still rising magnetic field and then oscillates for about three periods before it expands to the wall when the external field drops. As the field changes sign and increases again a new implosion begins and the compression process is repeated. The magnetic probe on the tube axis becomes luminous when the plasma reaches the centre at about $t = 1.5\ \mu$sec. One of the flute instabilities can be seen as a luminous streak separating from the lower half of the plasma at about this time and reaches the wall at $t = 2.5\ \mu$sec.

Fig. 3 shows framing camera pictures of a discharge which is similar to that of fig. 2 except that the initial pressure is greater and there is no magnetic probe present. The upper row of photographs (up to 2.40 μsecs) show successive stages of one discharge whilst the lower row consists of photographs selected from two other camera runs. The discharges were repeatable in their characteristics so that fig. 3 shows faithfully the progress of a typical discharge during the first half-cycle and the beginning of the second.

The cylindrical implosion shows up clearly and the plasma is seen to be concentrated in a cylindrical annulus



ig. 3 Framing camera photographs of a discharge in deuterium at an initial pressure of 150 microns taken using a high speed mera which viewed the discharge axially. The time in microseconds from the beginning of the first implosion is shown under ich photograph. The exposure time of each frame is 0.12 microseconds.

with an inner core of trapped axial magnetic field. The framing camera has a depth of field of about 10 cm and was focussed on the mid plane of the coil. For the first two microseconds the plasma is in sharp focus but becomes increasingly blurred towards the end of the half-cycle; this is because the plasma is being ejected from the ends of the coil. During the new implosion at the beginning of the next half-cycle the plasma is again in focus.

Irregularities in the plasma cylinder develop during the implosion phase and these subsequently become well-defined on the outer surface of the plasma and are believed to be instabilities of Rayleigh-Taylor type. The irregular inner surface of the quartz discharge tube provides nonuniform starting conditions and these perturbations grow in response to the inertial forces produced by the inward acceleration of the interface between plasma and magnetic field. The growth time of the instability is a fraction of a microsecond and the typical wavelength, or wavelength of "maximum instability," is about 1.5 cms. The shape of the instability—sharp flutes of dense fluid separated by round-ended troughs of the lighter fluid—is characteristic of the later stages of Rayleigh-Taylor instabilities.

The temperature of the plasma in fig. 3 has been estimated in a number of different ways. The shock velocity gives a temperature of 3×10^4 °K but this is a lower limit since at these temperatures joule heating is comparable with shock heating. Moreover estimates of the temperature from shock velocity are imprecise in this range where the temperature is a sensitive function of the shock velocity. The temperature has also been estimated from the measured damping of the radial oscillations [12] assuming that this damping is purely resistive; this method gives an estimated temperature of 6×10^4 °K. Pressure-balance calculations yield a temperature of 7×10^4 °K and a particle density of 10^{17} cm^{-3}. Thus none of these estimates is precise but they indicate that the temperature is of order 6×10^4 °K and the particle density of order 10^{17} cm^{-3}.

Growth Rate

The instability of an accelerated fluid interface was first studied by Rayleigh [13] and the theory subsequently developed by Taylor [14]. Experimentally, observations on liquid-liquid interfaces have been made by Lewis [15] and Allred and Blount [16] among others. If ϱ_1 and ϱ_2 are the densities of the lighter and heavier fluids respectively and a is an acceleration applied normally to the interface, then an initial disturbance of amplitude A_0 and wave number k ($k = 2\pi/\lambda$ where λ is the wavelength) is amplified in the form

$$A = A_0 e^{nt} \qquad (1)$$

where

$$n = \left[-\left(\frac{\varrho_2 - \varrho_1}{\varrho_2 + \varrho_1}\right) a k\right]^{\frac{1}{2}} \qquad (2)$$

The interface is unstable only when the acceleration is directed from the lighter to the heavier fluid and equation 2 shows that the growth rate increases indefinitely with increasing wave number (decreasing wavelength).

Kruskal and Schwarzschild [17] were the first to discuss Rayleigh-Taylor instabilities of an accelerated interface between an electrically conducting fluid and a magnetic field and they described in detail the way in which the inertial forces produce charge separation and particle drift which drive the instability. They showed that though the magnetic field produces the plasma motion the gross nature of the instability and its growth rate are the same as for ordinary fluids.

We now apply Taylor's theory to the observed instabilities though this comparison can only be approximate since Taylor's theory is a linear theory for an incompressible fluid of infinite extent and none of these idealizations apply in our case. Since we can regard the magnetic field as a fluid of zero density equation 2 becomes

$$n = (ak)^{\frac{1}{2}} = \left(\frac{2\pi a}{\lambda}\right)^{\frac{1}{2}} \qquad (3)$$

In our experiment λ is 1.5 cm and a is 5×10^{12} cms/sec^2. Thus the characteristic growth time is

$$\tau = \frac{1}{n} = \left(\frac{\lambda}{2\pi a}\right)^{\frac{1}{2}} = 2 \times 10^{-7} \text{ secs} \qquad (4)$$

which is of the order of the observed value.

Conditions in this experiment are particularly favourable for the growth of Rayleigh-Taylor instabilities because of the constant acceleration for a period of one microsecond, and since the instability grows like e^{nt} and n is proportional to $a^{1/2}$, the shorter the time taken to reach a final velocity i.e. the greater the impulsive nature of the acceleration, the less the growth of the disturbance. In the limit of a pure step-function in velocity there is no growth at all.

Damping Mechanism

Taylor's theory ignores effects such as surface tension and viscosity which will eliminate or damp out the shorter wavelengths. However, subsequent investigators [18]—[22] have considered these effects and shown that surface tension and viscosity act as damping processes which produce a wave number k_m of "maximum instability." The effect of viscosity, for example, increases with decreasing wavelength because the shearing forces increase, and the wavelength of maximum instability is essentially that wavelength at which the rate of doing work by inertial forces is equal to the rate of energy dissipation by viscous forces.

In a plasma the instabilities will be damped by the resistive diffusion of magnetic field into the plasma as well as by viscosity, and the inertial, viscous and resistive contributions to the growth rate n can be written:

Inertial term:	$(a k)^{1/2}$
Viscous term:	νk^2
Resistive term:	$k^2/4\pi\sigma$

where ν is the kinematic viscosity and σ is the electrical conductivity of the plasma. The inertial term represents the rate at which the instabilities grow in response to inertial forces whilst the other terms are the reciprocals of the characteristic times for momentum and magnetic field to diffuse across a wavelength and are obtained by observing the similarity of the diffusion equations for momentum and magnetic field, a similarity pointed out by Batchelor [23]. The relative importance of viscosity and resistivity as damping processes is given by the dimensionless parameter P where

$$P = 4\pi\sigma\nu = R_m/R_h \tag{5}$$

P is the ratio of the diffusivities of momentum and magnetic field and is also the ratio of hydromagnetic to hydrodynamic Reynolds number.

The conductivity and viscosity of a fully ionised plasma are given by Spitzer [24] and Marshall [25] and we may write for deuterium,

$$\sigma = \frac{1{\cdot}16 \cdot 2^{3/2} (kT_e)^{3/2}}{\pi^{3/2} \cdot m_e^{1/2} e^2 c^2 \ln \Lambda} \tag{6}$$

$$\nu = \frac{\mu}{n\, m_i} = \frac{5\, (kT_i)^{5/2}}{8\pi^{1/2} n\, m_i^{1/2} e^4 \ln \Lambda} \tag{7}$$

where T_e and T_i are the electron and ion temperatures, m_e and m_i the electron and ion masses, n is the ion density, e is the electronic charge, c the velocity of light, k is Boltzmann's constant, μ the dynamic viscosity and $\ln \Lambda$ is the logarithm of a collision integral tabulated by Spitzer which is approximately seven for our conditions. These equations show that the electrical conductivity depends on the electron temperature and the viscosity on the ion temperature though at our temperatures and densities relaxation times are so short (much less than a microsecond) that T_e and T_i are effectively the same. Moreover collision mean free paths are short compared with the Larmor radii so that it is permissible to use a simple scalar conductivity and viscosity.

Inserting numerical values for σ and ν gives

$$P = 4\pi\sigma\nu = \frac{3.2 \times 10^{\ 5}}{n} T^4 \tag{8}$$

showing that P is very sensitive to temperature. The damping effects of viscosity and resistivity will be approximately equal when $P = 1$ and for deuterium at a particle density of 10^{17} cm^{-3} this corresponds to a temperature of 2×10^5 °K. At temperatures above 2×10^5 °K the instabilities are damped primarily by viscosity and at lower temperatures by resistivity. Thus at our estimated temperature of 6×10^4 °K it is resistivity that determines the wavelength of greatest instability.

Equating the inertial and resistive terms in the growth rate gives

$$\lambda_m = (\pi/2)^{\frac{1}{3}}\, a^{-\frac{1}{3}}\, \sigma^{-\frac{2}{3}} \tag{9}$$

howing that λ_m is insensitive to a. Since σ is proportional o $T_e^{3/2}$, λ_m is inversely proportional to the electron emperature. Putting $\lambda_m = 1.5$ cms, $a = 5 \times 10^{12}$ cms/sec^2 and $\ln \Lambda = 7$ gives $T_e = 5.8 \times 10^4$ °K, in agreement vith the estimated value.

At higher temperatures where viscosity provides the damping we should expect

$$\lambda_m = 2\pi a^{-\frac{1}{3}}\, \nu^{\frac{2}{3}} \tag{10}$$

o that the wavelength is again insensitive to the acceleration but since ν varies as $T_i^{5/2}$, λ_m should vary as $T_i^{5/3}$ and should rise rapidly with increasing temperature.

Conclusions

Analysis of high speed photographs of the instabilities which develop during the compression of a plasma by a rapidly rising uniform axial magnetic field suggests that these are produced by inertial forces resulting from the inward acceleration of the boundary between magnetic field and plasma. The estimated temperature and density of the deuterium plasma is 6×10^4 °K and 10^{17} cm^{-3} and under these conditions the instabilities are damped by electrical resistivity. The agreement of the observed growth rate and the observed wavelength with simple theory tends to support this conclusion. In more powerful discharges the instabilities should be damped by viscosity and this increases rapidly with temperature. The growth of the instabilities would also be limited by a more impulsive acceleration.

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UN CRITERE NECESSAIRE DE STABILITE HYDROMAGNETIQUE POUR UN PLASMA EN SYMETRIE DE REVOLUTION

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Pour ce travail, on établit une condition nécessaire de stabilité pour un plasma en symétrie de révolution. Elle correspond à des déplacements localisés au voisinage des surfaces à pression constante sur lesquelles les lignes de force magnétique sont fermées. Ce critère se réduit à celui de Suydam en symétrie cylindrique.

Introduction

En géométrie cylindrique B. R. Suydam [1] a donné un critère de stabilité qui est en fait un critère vis-à-vis des déplacements limités au voisinage d'une surface $\varrho =$ constante, surface qui dans ce cas est un cylindre.

En effet nous exprimons le déplacement $\boldsymbol{\xi}$ sous la forme

$$\boldsymbol{\xi} = [\xi_r(r), \xi_\theta(r), \xi_z(r),]\, e^{ikz + im\theta}$$

On peut dans le principe variationnel donné par Bernstein et autres [2] intégrer sur θ et z et minimiser algébriquement sur ξ_θ et ξ_z et obtenir

$$\delta W_{km} = \int 2r\,dr \left\{ \frac{(rf\xi' + g\xi)^2}{m^2 + k^2 r^2} + (f^2 - h)\,\xi^2 \right]$$

$$\text{oú} \begin{cases} f = kB_z + (m/r)\,B_e \\ g = kB_z - (m/r)\,B_\theta \\ h = (2/r)\,J_z B_\theta \end{cases}$$

k et m étant convenablement choisis, $f(r)$ s'annulera au voisinage d'un cylindre quelconque dans le plasma; soit $r = a$ ce cylindre. Supposons un déplacement localisé à son voisinage. Nous pouvons alors remplacer r, $g(r)$ et $h(r)$ par a, $g(a)$, $h(a)$ et $f(r)$ par $(r - a)\,f'(a)$ et écrire en posant $t = r - a$

$$\delta W = 2a \int_{-\eta}^{\eta} \left[\frac{(atf'(a)\,\xi' + g(a)\,\xi)^2}{m^2 + k^2 a^2} - h(a)\,\xi^2 \right] dt$$

la perturbation étant limitée entre les 2 cylindres de rayon $r - \eta$, $r + \eta$ où η est très petit.

δW peut encore s'écrire en posant $t = \eta x$ et en intégrant par partie le terme en $\xi\,\xi'$:

$$\delta W = \frac{2a^3 f'^2(a)\,\eta}{m^2 + k^2 a^2} \int_{-1}^{+1} (x^2 \xi'^2 + A\xi^2)\,dx$$

avec

$$A = \frac{1}{a^2 f'^2}\,(g^2 - agf' - h\,(m^2 + k^2 a^2)) = \frac{2p'}{a B_z^2\,(\mu'/\mu)^2}$$

$$\mu = B_\theta / (r B_z)$$

Pour obtenir la dernière forme de A on a tenu compte de la relation d'équilibre du plasma:

$$p' + B_z B'_z + B_\theta B'_\theta + (B_\theta^2/r) = 0$$

Or on peut démontrer (voir annexe) que

$$\int_{-1}^{+1} (x^2 \xi'^2 + A\xi^2)\,dx \geq (A + {}^1/_4) \int_{-1}^{+1} \xi^2\,dx$$

et qu'il existe des déplacements qui permettent d'approcher d'aussi près que l'on veut l'égalité.

Ainsi pour avoir $\delta W > 0$, il faut

$$\frac{2p'}{B_z^2} + \frac{a}{4} \left(\frac{\mu'}{\mu} \right)^2 \geq 0$$

c'est l'expression de Suydam. Elle fournit ainsi un critère nécessaire et suffisant pour les déplacements localisés dans le cas de la géométrie cylindrique.

En géométrie de révolution la symétrie du problème étant moins forte, une seule intégration dans l'expression de δW peut être faite facilement. Pour continuer nous ferons alors l'hypothèse des déplacements localisés aux voisinages de surface à égale pression de type particulier correspondant au choix de Suydam en symétrie cylindrique.

Nous obtiendrons finalement un critère qui généralise le critère de Suydam, évidemment nécessaire car correspondant à des déplacements possibles, mais probablement pas suffisant en général pour la stabilité vis-à-vis de tous les déplacements suffisamment localisés autour d'une surface à pression constante.

Les équations fondamentales

L'équilibre d'un plasma à l'approximation hydromagnétique est déterminé par les équations suivantes:

$$\operatorname{div} \mathbf{B} = 0$$

$$\mathbf{B} \times \operatorname{rot} \mathbf{B} = - \operatorname{grad} p$$

En symétrie de révolution, il est commode d'écrire:

$$\mathbf{B} = T\,\frac{\mathbf{e}_\theta}{r} - \frac{\mathbf{e}_\theta}{r} \times \operatorname{grad} \psi \qquad (1)$$

qui est la solution générale dans ce cas de $\operatorname{div} \mathbf{B} = 0$

r = distance d'un point à l'axe de révolution

z = coordonnée le long de l'axe de révolution

$\mathbf{e}_\theta$ = vecteur unitaire perpendiculaire au plan méridien passant par ce point

p = pression du plasma (définie à 4π près)
T et ψ = sont des fonctions quelconques.

La deuxième équation d'équilibre fournit les renseignements suivants: T et p ne sont fonctions que de ψ, ellemême définie par l'équation suivante:

$$H\psi = -r^2 \frac{dp}{d\psi} - \frac{1}{2}\frac{dT^2}{d\psi} \tag{2}$$

avec

$$H = \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$$

Dans ce travail, nous étudions la stabilité des équilibres régis par ces équations.

Le point de départ sera le principe d'énergie développé par I. B. Bernstein et autres [2] que nous prendrons sous la forme:

$$\begin{aligned} \delta W_F = \frac{1}{2}\int d\tau \{ & [Q + (\mathbf{n}\cdot\boldsymbol{\xi})(\mathbf{j}\times\mathbf{n})]^2 \\ & + \gamma p (\operatorname{div}\boldsymbol{\xi})^2 - 2(\mathbf{n}\cdot\boldsymbol{\xi})^2(\mathbf{j}\times\mathbf{n})\cdot(\mathbf{B}\cdot\nabla)\,\mathbf{n}\} \end{aligned} \tag{3}$$

où $Q = \operatorname{rot}(\boldsymbol{\xi}\times\mathbf{B})$
$\mathbf{n}$ = est le vecteur unité normal à la surface p = constant
$\boldsymbol{\xi}$ = est un déplacement arbitraire, satisfaisant à la condition $\mathbf{n}\cdot\boldsymbol{\xi} = 0$ à la limite du plasma.

La stabilité pour les déplacements qui ne déforment pas la surface du plasma sera obtenue si δW_F reste toujours positif.

Le critère de stabilité

Nous emploierons le système de coordonnées orthogonal suivant, introduit dans [2], adapté à l'équilibre dont on recherche la stabilité: Les surfaces de coordonnées seront:

ψ = constantes, θ = constantes, χ = constantes

Les surfaces χ = constantes sont choisies normales aux surfaces ψ et θ de telle sorte que (ψ, θ, χ) forme un système orthogonal direct.

L'élément de volume s'écrit:

$$d\tau = J\,d\psi\,d\theta\,d\chi$$

Dans ces coordonnées, l'opérateur gradient s'écrit:

$$\nabla = rB\,\mathbf{e}_\psi\frac{\partial}{\partial\psi} + \frac{1}{r}\,\mathbf{e}_\theta\frac{\partial}{\partial\theta} + \frac{1}{JB}\,\mathbf{e}_\chi\frac{\partial}{\partial\chi}$$

B: longueur de la projection de $\mathbf{B}$ sur la plan méridien.

De l'équation d'équilibre et de l'équation (1) on tire:

$$\mathbf{J} = \operatorname{rot}\mathbf{B} = j\,\mathbf{e}_\theta + B\,T'\,\mathbf{e}_\chi$$

avec

$$j = -\frac{r}{J}\frac{\partial}{\partial\psi}(JB^2) \tag{4}$$

et

$$p' = \frac{j}{r} - \frac{TT'}{r^2} \tag{5}$$

$$p' \text{ et } T' \text{ signifie } \frac{dp}{d\psi} \text{ et } \frac{dT}{d\psi}$$

Un déplacement $\boldsymbol{\xi}$ est déterminé par ses composantes $(\xi_\psi \xi_\theta \xi_\chi)$

En posant
$rB\xi_\psi = X$, $\xi_\theta/r = Y$, $\xi_\chi/B = Z$ et $Y - (T/r^2)Z = U$

et l'opérateur

$$J\mathbf{B}\cdot\nabla = \frac{\partial}{\partial\chi} + \frac{JT}{r^2}\frac{\partial}{\partial\theta} = D$$

L'équation (3) s'écrit:

$$\begin{aligned} \delta W_F = \frac{1}{2}\int d\psi\,d\theta\,d\chi \Big\{ & \frac{1}{Jr^2B^2}[DX]^2 \\ & + \frac{r^2}{J}\left[\frac{\partial U}{\partial\chi} - T\frac{\partial}{\partial\psi}\left(\frac{JX}{T^2}\right)\right]^2 + JB^2\left[\frac{\partial U}{\partial\theta} + \frac{\partial X}{\partial\psi} + \frac{j}{rB^2}X\right]^2 \\ & + \frac{\gamma p}{J}\left[\frac{\partial}{\partial\psi}(JX) + J\frac{\partial U}{\partial\theta} + DZ\right]^2 - 2JKX^2 \end{aligned} \tag{6}$$

avec

$$\begin{aligned} K &= \frac{TT'}{r^2}\frac{\partial(\log r)}{\partial\psi} - \frac{j}{r}\frac{\partial(\log BJ)}{\partial\psi} \\ &= -\frac{TT'}{r^2}\frac{\partial[\log(BJ/r)]}{\partial\psi} - p'\frac{\partial(\log BJ)}{\partial\psi} \end{aligned} \tag{7}$$

Remarquons que $K < 0$ est une condition suffisante de stabilité.

Cette condition, dans le cas de symétrie cylindrique $(B = B(r), J = 1/B, \psi = \psi(r))$ s'écrit $TT' < 0$ condition bien connue dans ce cas.

Posons:

$$\begin{aligned} X &= \sum_m X_1^m \cos m\theta + X_2^m \sin m\theta \\ U &= \sum_m (U_1^m/m)\sin m\theta - (U_2^m/m)\cos m\theta \\ Z &= \sum_m Z_1^m \cos m\theta + Z_2^m \sin m\theta \end{aligned} \tag{8}$$

On obtient pour chaque m $(m \neq 0)$

$$\begin{aligned} (\delta W_F)_m = \frac{\pi}{2}\int d\psi\,d\chi \Big\{ & \Big[\left(\frac{\partial X_1}{\partial\chi} + m\frac{JT}{r^2}X_2\right)^2 \\ & + \left(\frac{\partial X_2}{\partial\chi} - m\frac{JT}{r^2}X_1\right)^2\Big]\frac{1}{Jr^2B^2} \\ & + \frac{r^2}{J}\left[\frac{1}{m}\frac{\partial U_1}{\partial\chi} - T\frac{\partial}{\partial\psi}\frac{JX_2}{r^2}\right]^2 \\ & + \frac{r^2}{J}\left[\frac{1}{m}\frac{\partial U_2}{\partial\chi^1} + T\frac{\partial}{\partial\psi}\frac{JX_1}{r^2}\right]^2 \\ & + JB^2\left[U_1 + \frac{\partial X_1}{\partial\psi} + \frac{j}{rB^2}X_1\right]^2 \\ & + JB^2\left[\mathrm{U}_2 + \frac{\partial X_2}{\partial\psi} + \frac{j}{rB^2}\mathrm{X}_2\right]^2 \\ & + \frac{\gamma p}{J}\left[\frac{\partial}{\partial\psi}(JX_1) + JU_1 + \frac{\partial Z_1}{\partial\chi} + m\frac{JT}{r^2}Z_2\right]^2 \\ & + \frac{\gamma p}{J}\left[\frac{\partial}{\partial\psi}(JX_2) + JU_2 + \frac{\partial Z_2}{\partial\chi} - m\frac{JT}{r^2}Z_1\right]^2 \\ - 2J(X_1^2 + X_2^2) & \left(\frac{\partial}{\partial\psi}(\log r)\frac{TT'}{r^2} - \frac{j}{r}\frac{\partial}{\partial\psi}(\log BJ)\right)\Big\} \end{aligned}$$

L'indice m a été supprimé sur $X_1X_2\,U_1U_2\,Z_1Z_2$ pour simplifier.

Posons

$$\begin{aligned} X_m &= X_1^m + iX_2^m \\ U_m &= U_1^m + iU_2^m \\ Z_m &= Z_1^m + iZ_2^m \end{aligned}$$

L'indice m de $X\,U\,Z$ sera supprimé chaque fois qu'il n'y aura pas de risque de confusion. On peut écrire

$$(\delta W_F)_m = \frac{\pi}{2}\int d\psi\, d\chi \left\{ \frac{1}{Jr^2 B^2}\left|\frac{\partial X}{\partial \chi} - i m \frac{JT}{r^2} X\right|^2 + \frac{r^2}{J}\left|\frac{1}{m}\frac{\partial U}{\partial \chi} + iT \frac{\partial}{\partial \psi}\left(\frac{JX}{r^2}\right)\right|^2 + JB^2\left|U + \frac{\partial X}{\partial \psi} + \frac{j}{rB^2} X\right|^2 + \frac{\gamma p}{J}\left|\frac{\partial}{\partial \psi}(JX) + JU + \frac{\partial Z}{\partial \chi} - i m \frac{JTZ}{r^2}\right|^2 - 2J\left(\frac{\partial}{\partial \psi}(\log r)\frac{TT'}{r^2} - \frac{j}{r}\frac{\partial}{\partial \psi}(\log BJ)\right)\left|X\right|^2 \right\} \qquad (9)$$

Dans la suite nous utiliserons le résultat suivant relatif aux solutions périodiques d'une équation différentielle du type suivant:

$$\frac{\partial Z}{\partial \chi} + i\,a(\chi)\, Z + b(\chi) = 0$$

On suppose que $a(\chi)$ et $b(\chi)$ sont des fonctions de période χ_0. Si

$$\oint a(\chi)\, d\chi = \int_{\chi}^{\chi+\chi_0} a(x)\, dx \neq 2K\pi \ (K \text{ entier})$$

on a une solution périodique, de période χ_0, et une seule

$$Z(\chi) = \frac{1}{1 - \exp i \oint a(x)\, dx}\int_{\chi}^{\chi+\chi_0} b(x) \exp i \int_{\chi}^{x} a(u)\, du\, dx \qquad (10)$$

En effet, on peut vérifier directement que c'est *une solution. Elle est périodique*. En effet:

$$Z(\chi + \chi_0) = \frac{1}{1 - \exp i \oint a(x)\, dx}\int_{\chi+\chi_0}^{\chi+2\chi_0} b(x) \exp i \int_{\chi+\chi_0}^{x} a(u)\, du\, dx$$

et faisons le changement de variable

$$x = \chi_0 + t$$
$$u = \chi_0 + v$$

$$Z(\chi + \chi_0) = \frac{1}{1 - \exp i \oint a(x)\, dx}\int_{\chi}^{\chi+\chi_0} b(t) \exp i \int_{\chi}^{t} a(v)\, dv\, dt = Z(\chi)$$

Enfin il n'y a en pas d'autre car on obtient la solution générale en ajoutant à cette fonction périodique particulière une solution générale de l'équation homogène $(\partial z / \partial \chi) + i\, a(\chi)\, z = 0$, c'est-à-dire

$$z = C \exp\left\{-i \int_{\chi_1}^{\chi} a(x)\, dx\right\}$$

solutions qui ne sont jamais périodiques.

Revenons à $(\delta W_F)_m$ et minimisons d'abord sur Z. En posant

$$f = \frac{\gamma p}{J}\left(\frac{\partial}{\partial \psi}(JX) + JU + \frac{\partial Z}{\partial \chi} - i m \frac{JT}{r^2} Z\right)$$

l'équation d'Euler s'écrit:

$$\frac{\partial f}{\partial \chi} = i m \frac{TJ}{r^2} f$$

Soit $f = f(\psi) \exp\left[i m \int_0^{\chi} (JT/r^2)\, d\chi\right]$

f doit être périodique en χ (période χ_0). Ceci ne peut se produire que si

$$\oint (JT/r^2)\, d\chi = 2k\pi$$

$\oint$ indique l'intégrale prise sur une période.

Sauf pour un ensemble de points particuliers (de mesure nulle) cette relation n'est pas vérifiée. Donc $f(\psi) = 0$. La minimisation impose donc que $f = 0$.
Encore faut-il trouver une solution Z convenable c'est-à-dire périodique en χ déterminée par $f = 0$. Or $f = 0$ est de la forme étudiée ci-dessus

$$\frac{\partial Z}{\partial \chi} - i m \frac{JT}{r^2} Z + JU + \frac{\partial}{\partial \psi}(JX) = 0$$

avec

$$a(\chi) = -m JT/r^2$$

$$b(\chi) = JU + \frac{\partial}{\partial \psi} JX$$

fonction périodique de période χ_0 et

$$\oint a(x)\, dx = -m \oint (JT/r^2) \neq 2k\pi \text{ en général}$$

La solution cherchée qui donne le minimum par rapport à la variable Z est donc

$$Z = \frac{1}{1 - \exp\left[-im \oint (JT/r^2)\, d\chi\right]} \int_{\chi}^{\chi+\chi_0}\left\{\left(JU + \frac{\partial}{\partial \psi} JX\right) \exp\left[-im \int_{\chi}^{x} (JT/r^2)\, du\right]\right\} dx \qquad (11)$$

Interprétation de $\oint (JT/r^2)\, d\chi$.

La ligne magnétique tracée sur la surface ψ = constante et qui passe en un point M de cette surface a pour équation:

$$J d\chi = \frac{r\, d\theta}{B_\chi} = \frac{r^2}{T} d\theta$$

Soit:

$$d\theta = (JT/r^2)\, d\chi$$
$$\theta_0 = \oint (JT/r^2)\, d\chi \qquad (12)$$

Ainsi la quantité $\oint (JT/r^2)\, d\chi$ n'est autre chose que l'angle θ pris entre deux point M et M', M' étant le point où la ligne magnétique recoupe pour la première fois le parallèle passant par M. Cet angle est indépendant du point M.

Les cas particuliers où $\theta_0 = 2k\pi$ signifient que la ligne magnétique se referme. En général $\theta_0 \neq 2k\pi$ et la ligne magnétique ne se referme pas.

Posons alors

$$\eta = \frac{\partial X}{\partial \psi} + U$$

$$\varepsilon = \frac{\partial X}{\partial \chi} - \frac{imJT}{r^2} X \qquad (13)$$

$$\xi = \frac{i}{m}\frac{\partial U}{\partial X} - T\frac{\partial}{\partial \psi}\left(\frac{J}{r^2} X\right) - \frac{JT'}{r^2} X$$

Entre les $\eta\, \varepsilon\, \xi$ il existe la relation:

$$\frac{\partial \eta}{\partial \chi} - \frac{\partial \varepsilon}{\partial \psi} + im\,\xi = 0 \qquad (14)$$

Moyennant cette relation, on peut tirer univoquement X et U à partir $\eta\ \varepsilon\ \xi$ périodiques en χ en utilisant une fois encore le théorème (10) appliqué à l'équation:

$$\frac{\partial X}{\partial \chi} - \frac{imJT}{r^2} X = \varepsilon$$

On peut alors écrire:

$$\delta W_{Fm} = \frac{\pi}{2}\int d\psi d\chi \left\{ \frac{|\varepsilon|^2}{Jr^2B^2} + \frac{r^2}{J}\left[\xi + \frac{JT'}{r^2} X\right]^2 + JB^2\left[\eta + \frac{j}{rB^2} X\right]^2 - 2J|X|^2 K \right\} \tag{15}$$

avec:

$$X = \frac{-1}{1 - \exp\left[-im \oint (JT/r^2)\, d\chi\right]} \int_{\chi}^{\chi+\chi_0} \varepsilon \exp\left[-im\int_{\chi}^{x} (JT/r^2)\, du\right] dx$$

Remarquons alors que si on pose:

$$\theta(\chi, a) = \int_a^{\chi} (JT/r^2)\, du$$

on a:

$$\theta(\chi + \chi_0, a) = \theta(\chi, a) + \theta_0$$

donc:

$$\theta(\chi, a) = p(\chi, a) + \theta_0\, \chi/\chi_0$$

avec:

$$p(\chi + \chi_0, a) = p(\chi, a)$$

ce qui permet d'écrire:

$$X = \frac{-\exp\left[imp(\chi,a) + im\theta_0\, \chi/\chi_0\right]}{-\exp(-im\theta_0)} \int \varepsilon(x)\, e^{-imp(x,a)}\, e^{-im\theta_0 x/\chi_0}\, dx \tag{16}$$

Comme $\varepsilon(x)\, e^{-imp(x,a)}$ est une fonction périodique en x (période χ_0) on peut développer cette expression comme suit:

$$\varepsilon(x)\, e^{-imp(x,a)} = \sum_k a_{km}\, e^{2i\pi kx/\chi_0} \tag{17}$$

et en reportant, on trouve:

$$X = \chi_0 \sum_k \frac{a_{km}}{i(2\pi k - m\theta_0)}\, e^{imp(\chi,a)}\, e^{2i\pi k\chi/\chi_0} \tag{18}$$

Remarquons que le changement du point de référence a se traduirait par un simple changement des a_{km}. Pour la suite, nous écrirons $p(\chi)$.

En analogie avec l'hypothèse de Suydam [1], nous étudierons les déplacements (k, m) tels que $2\pi k - m\theta_0$ est voisin de 0.

Soit une surface $\psi =$ constante telle que $\theta_0/2M$ ne soit pas rationnel. On peut toujours choisir les entiers k et m de telle sorte que $2\pi k - m\theta_0$ soit aussi voisin de 0 que l'on veut.

On peut supposer que ces déplacements sont parmi les plus dangereux. Ils coïncident d'ailleurs avec les déplacements de Suydam si l'on se place dans le cas particulier de la symétrie cylindrique.

Aussi allons-nous étudier les déplacements définis par

$$\begin{aligned} \varepsilon &= a_{km}\, e^{imp(\chi)}\, e^{2i\pi k\chi/\chi_0} \\ X &= p_{km} \frac{\chi_0}{i(2k - m\theta_0)}\, e^{imp(\chi)}\, e^{2i\pi k\chi/\chi_0} \end{aligned} \tag{19}$$

On constate que:

$$X = \frac{-i\chi_0}{2\pi k - m\theta_0}\, \varepsilon = \frac{-i}{\nu}\, \varepsilon$$

avec:

$$\nu = \frac{2\pi k - m\theta_0}{\chi_0}$$

Posons alors:

$$X = \varrho e^{i\varphi} \qquad \varepsilon = i\nu X = i\nu\varrho e^{i\varphi}$$

Donc:

$$\varrho = \varrho(\psi)$$

$$\varphi = mp(\chi) + 2\pi k \frac{\chi}{\chi_0} + C(\psi) = m\theta(\chi, a) + \nu\chi + C(\psi)$$

$C(\psi)$ étant une constante arbitraire.

Introduisons la fonction b telle que: $\eta = bX$

$$\begin{aligned} \frac{\partial \eta}{\partial \chi} &= \frac{\partial b}{\partial \chi} X + b\left(i\nu X + \frac{imJT}{r^2} X\right) \\ &= X\left(\frac{\partial b}{\partial \chi} + i\nu b + \frac{imJT}{r^2} b\right) \end{aligned}$$

$$\frac{\partial \varepsilon}{\partial \psi} = X\left[i\frac{\partial \nu}{\partial \psi} + i\nu\frac{\varrho'}{\varrho} - \nu\frac{\partial \varphi}{\partial \psi}\right]$$

D'après la relation (14):

$$im\xi = \frac{\partial \varepsilon}{\partial \psi} - \frac{\partial \eta}{\partial \chi} = X\left[i\frac{\partial \nu}{\partial \psi} + i\nu\frac{\varrho'}{\varrho} - \nu\frac{\partial \varphi}{\partial \psi} - \frac{\partial b}{\partial \chi} - i\nu b - \frac{imJT}{r^2} b\right]$$

$$\xi + \frac{JT'}{r^2} X = X\left[\frac{JT'}{r^2} + \frac{1}{m}\frac{\partial \nu}{\partial \psi} + \frac{\nu}{m}\frac{\varrho'}{\varrho} + \frac{i\nu}{m}\frac{\partial \varphi}{\partial \psi} + \frac{i}{m}\frac{\partial b}{\partial \chi} - \frac{\nu}{m} b - \frac{JT}{r^2} b\right]$$

$$\eta + \frac{j}{rB^2} X = X\left(b + \frac{j}{rB^2}\right)$$

D'où:

$$\delta W_{Fm} = \frac{\pi}{2}\int d\psi d\chi\, \varrho^2 \left\{\frac{\nu^2}{Jr^2B^2} + \frac{r^2}{J}\left[\frac{JT'}{r^2} + \frac{1}{m}\frac{\partial \nu}{\partial \psi} + \frac{i\nu}{m}\frac{\partial \varphi}{\partial \psi} + \frac{i}{m}\frac{\partial b}{\partial \chi} - \frac{\nu}{m} b - \frac{JT}{r^2} b + \frac{\nu}{m}\frac{\varrho'}{\varrho}\right]^2 + JB^2\left[b + \frac{j}{rB^2}\right]^2 - 2JK\right\} \tag{20}$$

Les calculs qui ont permis de passer de l'expression (9) de δW_{Fm} à l'expression (20) sont valables en dehors des surfaces $\psi =$ const. sur lesquelles les lignes de force sont fermées. Pour un m donné ces surfaces correspondent à des valeurs discrètes de ψ. L'équivalence entre (9) et (20) est donc établie si l'on exclut de l'intégration sur ψ un petit intervalle autour de chaque valeur singulière de ψ. Mais comme les quantités à intégrer dans (9) et (20) sont continues en ψ, les expressions (9) et (20) sont aussi égales sans restriction sur l'intégration en ψ. L'elimination du terme provenant de $\gamma p\, (\mathrm{div}\, \xi)^2$ nécessite une étude plus détaillée an voisinage des surfaces singulières de ψ.

Nous allons chercher un critère local, c'est-à-dire que nous nous limiterons aux déplacements localisés au voisinage des surfaces $\psi = \psi_0$ et tels que ν soit voisin de 0.

$$\nu(\psi) = (\psi - \psi_0)\left(\frac{\partial \nu}{\partial \psi}\right)_0 = (\psi - \psi_0)\,\nu'_0 \qquad (21)$$

Ce choix de déplacement est analogue au choix fait par Suydam. En effet, le critère de Suydam est en fait un critère local au voisinage des surfaces $r = r_0$ où ν cylindrique $= k + m B_\theta / r B_z$ (voir introduction).

Ce ν cylindrique correspond exactement au ν défini ci-dessus en symétrie de révolution. Les cylindres $r = r_0$ définis par Suydam se généralisent selon les surfaces $\psi = \psi_0$ où les lignes de champ magnétique sont fermées.

Ainsi dans δW_F nous remplacerons ν par $(\psi - \psi_0)\,\nu'_0$ et les quantités telles que r, B etc. … relatives à la solution d'équilibre seront prises constantes et égales à la valeur qu'elles prennent pour $\psi = \psi_0$. Elles dépendront encore en général de χ.

$$\delta W_F = \frac{\pi}{2}\int d\psi\, d\chi\, \varrho^2 \left\{ \frac{r^2}{J}\left[\frac{JT'}{r^2} + \frac{\nu_0'}{m} + (\psi - \psi_0)\frac{\nu_0'}{m}\frac{\varrho'}{\varrho} + \frac{i\,\partial h}{m\,\partial\chi} - \frac{JT}{r^2}h\right]^2 + JB^2\left[h + \frac{j}{rB^2}\right]^2 - 2JK \right\} \qquad (22)$$

Nous limiterons encore les déplacements en supposant la fonction h réelle. Dans ce cas le seul terme contenant $\partial h/\partial\chi$ dans l'intégrant est de la forme $\frac{1}{m^2}\left(\frac{\partial h}{\partial \chi}\right)^2$ et il est alors évident que l'expression sera minimum pour $m = \infty$; $\delta W_{F\infty}$ ne contiendra que la fonction h ce qui permet de minimiser par rapport à elle sans difficulté.

On trouve:

$$\delta W_{F\infty} = \frac{\pi}{2}\int d\psi\, d\chi \left\{ \frac{B^2 \varrho^2}{B^2 + \frac{T^2}{r^2}} \left| \frac{r^2}{J}\left(\frac{JT'}{r^2} + \frac{\nu_0'}{m} + (\psi - \psi_0)\frac{\nu'_0}{m}\frac{\varrho'}{\varrho}\right)^2 + \frac{2jT}{rB^2}\left(\frac{JT'}{r^2} + \frac{\nu_0'}{m} + (\psi - \psi_0)\frac{\nu_0'}{m}\frac{\varrho'}{\varrho}\right) + \frac{Jj^2T^2}{r^4B^4} \right| - 2JK\varrho^2 \right\} \qquad (23)$$

En intégrant par partie le terme $\varrho\,\varrho'$ par rapport à ψ, on peut écrire $\delta W_{F\infty}$ sous la forme $\delta W_{F\infty} = C^2 R$

$$R = \int d\psi\,[(\psi - \psi_0)^2 \varrho'^2 + A\varrho^2] \qquad (24)$$

avec

$$C^2 = \frac{\pi}{2}\left(\frac{\nu'_0}{m}\right)^2 \oint \frac{r^2}{J}\frac{B^2}{B^2 + \frac{T^2}{r^2}}\,d\chi$$

$$A = \oint \left\{ \frac{B^2 r^2 \left[\left(T' + \frac{jT}{rB^4}\right)\frac{J}{r^2} + \frac{\nu'_0}{2m}\right]^2}{J C^2 [B^2 + (T^2/r^2)]} - \frac{2JK}{C^2} \right\} d\chi - \frac{1}{4}$$

Or, on a démontré (voir annexe) que pour $\varrho(1) = \varrho(-1) = 0$ et $\varrho\,\varrho'$ continues

$$\int_{-1}^{+1} (\chi^2 \varrho'^2 + A\varrho^2)\,d\chi > \left(A + \frac{1}{4}\right)\int_{-1}^{+1} \varrho^2\,dx$$

et qu'il existait des déplacements qui rendent la valeur du première membre aussi voisine que l'on veut de alle du second.

On peut appliquer ce résultat à l'expression $\delta W_{F\infty}$ et on obtient le critère suivant:

$$\mathcal{Q}_\infty = \oint \left\{ \frac{B^2 r^2 \left[\left(T' + \frac{jT}{rB^4}\right)\frac{J}{r^2} + \frac{\nu_0'}{2m}\right]^2}{J[B^2 + (T^2/r^2)]} - 2JK \right\} d\chi > 0$$

En tenant compte de la formule (4), $\mathcal{Q}_\infty$ s'écrit encore:

$$\mathcal{Q}_\infty = \oint d\chi \left| \frac{r^2 B^2}{J\left(B^2 + \frac{T^2}{r^2}\right)} \left(\frac{J^2 B^2}{r^2}\frac{\partial}{\partial\psi}\left(\frac{T}{JB^2}\right) + \frac{\nu_0'}{2m}\right)^2 - 2JK \right| \qquad (25)$$

Dans cette expression

$$\frac{\nu_0'}{m} = -\frac{1}{\oint d\chi}\cdot\frac{\partial}{\partial\psi}\oint \frac{JT}{r^2}\,d\chi = -\bar{\mu}'$$

en posant
$$\mu = \frac{JT}{r^2}$$

$$\mu' = \frac{\partial}{\partial\psi}\frac{JT}{r^2} \qquad\qquad \bar{\mu} = \frac{\oint \mu\, d\chi}{\oint d\chi}$$

et en tenant compte des équations d'équilibre du système, on peut réécrire $\mathcal{Q}_\infty$ sous une autre forme:

$$\begin{aligned}\mathcal{Q}_\infty = {} & p' \oint \frac{\partial J}{\partial \psi}\,d\chi - p'^2 \oint \frac{J\,d\chi}{B^2[1 + (T^2/r^2B^2)]} \\ & + \frac{1}{4}T^2 \bar{\mu}'^2 \oint \frac{J\,d\chi}{r^2\mu^2[1 + (T^2/r^2B^2)]} \\ & - T^2 p' \bar{\mu}' \oint \frac{J\,d\chi}{r^2 B^2 \mu [1 + (T^2/r^2B^2)]}\end{aligned} \qquad (26)$$

Dans ces expressions, il reste encore une fonction arbitraire, c'est le paramétrage des courbes $\psi =$ constante, c'est-à-dire que l'on peut choisir arbitrairement pour χ une fonction quelconque de χ_0.

$$\chi = f(\chi_0)$$

χ_0 étant tel que:

$$J\,d\chi = J_0\,d\chi_0 \qquad (27)$$

Il nous reste à minimiser sur cette fonction f soumis à la restriction:

$$\lambda = \oint d\chi = \oint f'(\chi_0)\,d\chi_0 \qquad (28)$$

Dans l'expression (26) seuls les deux derniers termes varient dans la transformation $\chi = f(\chi_0)$. La minimisation est algébrique et la valeur minimum pour ces deux termes s'écrit:

$$\frac{1}{4\oint [1 + (T^2/r^2B^2)]\,(J/r^2)\,d\chi} \cdot \left[\frac{\partial}{\partial\psi}\oint \frac{JT}{r^2}\,d\chi - 2p'T\oint \frac{J\,d\chi}{r^2B^2}\right] - p'^2 T^2 \oint \frac{J\,d\chi}{r^2 B^4 [1 + (T^2/r^2B^2)]}$$

L'expression finale minimum de $\mathcal{Q}_\infty$ s'écrit:

$$\mathcal{Q}_\infty = \frac{1}{4\oint\left(1 + \frac{T^2}{r^2B^2}\right)\frac{J\,d\chi}{r^2}}\left[\frac{\partial}{\partial\psi}\oint \frac{JT}{r^2}\,d\chi - 2p'T\oint \frac{J\,d\chi}{r^2B^2}\right]^2 + p'\oint \frac{\partial J}{\partial\psi}\,d\chi - p'^2 \oint \frac{J\,d\chi}{B^2} \qquad (29)$$

Notons que $J\,d\chi = ds/B$, s designant l'abscisse curviligne des courbes à pression constante.

Le critère obtenu s'exprimera donc par: $Q_\infty > 0$.

Si nous nous plaçons en symétrie cylindrique on a $J = 1/B_z$, $T = r\,B_\theta$, $d\chi$ devient dz, $\partial/\partial\psi$ devient $\partial/\partial r$ et $\mu = JT/r^2 = B_0/(rB_z)$. Toutes les fonctions ne dépendent que de r. On obtient dans ce cas le critère de Suydam en utilisant l'équation d'équilibre

$$\frac{\mu'^2}{4} + \frac{2\,p'\mu^2}{r\,B^2} > 0, \tag{30}$$

p' *et* μ' signifiant alors $\frac{dp}{dr}$ et $\frac{d\mu}{dr}$

Note

Ce travail terminé, nous avons reçu un article de B. B. Kadomtsev [3] dans lequel l'auteur généralise aussi le critère de Suydam pour une géométrie torique quelconque mais en imposant à l'avance une forme particulière de déplacement localisé. Le résultat obtenu revient dans notre méthode à supposer b = constante. Dans ce cas nous retrouvons en géométrie de révolution son résultat. L'expression (29) est différente et plus sévère car nous avons supposé seulement que b est réelle et minimisé dans ce cas.

Le critère de B. B. Kadomtsev s'écrit avec nos notations:

$$1 \qquad\qquad 1$$

$$\frac{1}{4}\oint\frac{J\,d\chi}{r^2}\left(1+\frac{\oint(JT^2/r^2)\,d\chi}{\oint JB^2\,d\chi}\right)$$

$$\left[\frac{\partial}{\partial\psi}\oint\frac{JT}{r^2}\,d\chi - 2p'T\,\frac{\oint\frac{J\,d\chi}{r^2}\oint J\,d\chi}{\oint JB^2\,d\chi}\right]^2$$

$$+\,p'\oint\frac{\partial J}{\partial\psi}\,d\chi - p'^2\,\frac{(\oint J\,d\chi)^2}{\oint JB^2\,d\chi} > 0$$

Annexe

Il s'agit de démontrer que la quantité

$$S(y) = \frac{\int_{-1}^{+1} x^2\,y'^2\,dx}{\int_{-1}^{+1} y^2\,dx}$$

est toujours supérieure à 1/4 pour toutes fonctions y s'annulant à ± 1 continues ainsi que leurs dérivées, la valeur 1/4 pouvant être approchée d'aussi près que l'on veut par une fonction de cette classe (C).

Nous indiquerons les grandes lignes d'une démonstration basée sur le développement de y en polynomes de Legendre, orthogonaux sur $(-1, +1)$. Prenons pour y une combinaison linéaire des N premiers polynomes $P_n(x)$

$$y = \sum_1^N a_n P_n(x) \tag{A 1}$$

$$y(-1) = \sum_1^N (-1)^n a_n = 0$$

$$y(+1) = \sum_1^N a_n = 0$$

D'où

$$\sum_{n\;1,3,5\ldots}^{N} a_n = 0 \qquad\qquad \sum_{n\,=\,0,2,4\ldots}^{N} a_n = 0$$

En utilisant les propriétés intégrales des polynomes de Legendre on peut écrire:

$$S = \frac{1}{4} + \frac{\sum\limits_{n=1}^{N}(n^2-\frac{1}{2})\,a_n{}^2 + 2\sum\limits_{n<m}^{(n-m=0,2,4\ldots)} a_n\,a_m\,n\,(n+1)}{\Sigma\,2\,a_n{}^2/(2n+1)}$$

$$= \frac{1}{4} + \frac{R_N}{D_N}$$

$$R_N = R_{Np} + R_{Ni}\begin{cases} R_{N_p} \text{ n'ayant que des } n \text{ pairs} \\ R_{Ni} \text{ n'ayant que des } n \text{ impairs.}\end{cases}$$

La suite traite R_{Np}, le calcul pour R_{Ni} est identique.

En tenant compte de $\sum a_n = 0$ on peut poser:

$$a_0 = x_2$$
$$a_2 = x_4 - x_2$$
$$a_n = x_{n+2} - x_n$$
$$a_N = -x_N$$

R_N s'écrit alors:

$$R_N = \sum_{n=2}^{N}(2\,n-1)\,x_n^2 + \sum_{n=2}^{N-2}(2\,n+1)\,x_n\,x_{n+2}$$

forme quadratique que l'on peut décomposer ainsi:

$$R_N = \sum_{n=2}^{N-2}\alpha_n\left[x_n + \frac{2n+1}{2\alpha_n}\,x_{n+2}\right]^2 + \alpha_N\,x_N{}^2$$

avec $\alpha_2 = 3$ et

$$\alpha_n = 2n - 1 - \frac{(2n-3)^2}{2^2\alpha_{n-2}}$$

ce qui fournit:

$$\alpha_n = \frac{2\,n+1}{2} + \frac{1}{2\left(1+\frac{1}{5}+\cdots\cdots\frac{1}{2\,n-3}\right)}$$

α_n est donc toujours positif ce qui prouve que R_N l'est toujours aussi.

Ainsi: $S > \frac{1}{4}$ pour tout y de la forme (A1). S'il existait une fonction de la classe (C) telle que $S\,(y_1) < \frac{1}{4}$ on pourrait trouver un y de la forme (A 1) tel que $[S\,(y) - S\,(y_1)]$ soit arbitrairement petit, donc tel que $S\,(y) < \frac{1}{4}$ ce qui est impossible.

Il en résulte que $S\,(y) \geq \frac{1}{4}$ pour tous les y de la classe (C).

Pour montrer que l'on peut approcher d'aussi près que l'on veut la borne $\frac{1}{4}$, choisissons pour les x_n la forme suivante:

$$x_n = (-1)^{n/2}\left(\frac{1}{n^P} - \frac{1}{N^P}\right)t_n$$

avec $P > 0$ et

$$\frac{t_n+2}{t_n} = \frac{2n+1}{2\alpha_n} \text{ avec } t_2 > 0.$$

Alors

$$R_N = \sum_2^{N-2}\alpha_n t_n{}^2\left(\frac{1}{n^P} - \frac{1}{(n+2)^P}\right)^2 < 8\,P^2\sum_2^{N-2}\frac{t_n{}^2}{n^{2P+1}}$$

car $\alpha_n < 2\,n$

D'autre part:

$$D_n = \sum a_n{}^2\,\frac{2}{2n+1}$$

$$= 2X_2^2 + \sum_{n=2}^{N-2} \frac{2}{2n+1} t_n^2$$

$$\left[\frac{1}{n^P} + \frac{1}{N^P} + \frac{t_{n+2}}{t_n}\left(\frac{1}{(n+2)^P} - \frac{1}{N^P}\right)\right]^2$$

$$D_N > \frac{2}{3} \sum_{n=2}^{N-2} \frac{t_n^2}{n} \left(\frac{1}{n^P} - \frac{1}{N^P}\right)^2 \text{ car } 2n+1 < 3n$$

$$D_N > \frac{2}{3} \left| \sum_{n=2}^{N-2} \frac{t_n^2}{n^{2p+1}} - \frac{2}{N^p} \sum_{2}^{N=2} \frac{t_n^2}{n^{p+1}} \right|$$

Or $$\frac{2}{N^P} \sum_{2}^{N-2} \frac{t_n^2}{n^{P+1}} < \frac{2t_2^2}{N^P} \sum_{1}^{\infty} \frac{1}{n^{P+1}} = \frac{2t_2^2}{N^P} \xi(P+1)$$

ξ = fonction de Rieman.
Or on sait que

$$\xi(P+1) = \frac{1}{P} + \text{fonction entière}$$

D'où

$$\frac{2}{N^P} \sum_{n=2}^{N-2} \frac{t_n^2}{n^{P+1}} < \frac{2t_2^2}{PN^P} + \frac{2t_2^2}{N^P} \times \text{fonction entière} = K(N)$$

Ainsi pour $P > 0$, $k(N)$ tend vers zéro lorsque $N \to +\infty$

$$\frac{R_N}{D_N} < 12 P^2 \frac{1}{1 - K(N) \left\{\sum_{n=2}^{N-2} t_n^2 / n^{2P+1}\right\}^{-1}}$$

Donc R_N/D_N pour N assez grand sera inférieur à disons $24\,P^2$ et l'on peut donc déterminer des fonctions y telles que R_N/D_N soit aussi voisin de 0 que l'on veut en choisissant $P > 0$ assez petit.

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RAPID COMPRESSION HEATING OF A PLASMA IN THE LINEAR Z-PINCH

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The electrical characteristics of discharges with high and low initial rates of current rise are briefly described and the electrical measurements are used to determine the work done on the plasma by rapid compression. The particle and the magnetic energy in the discharge are discussed for both high and low initial rates of current rise (dI/dt (0)) and for discharges with and without an axial magnetic field (H_z). It is shown that the gas is heated much more efficiently both with and without H_z at high dI/dt (0). For dI/dt (0) $\sim 1.5 \cdot 10^{12}$ amps/sec ($H_z = 0$) 15% of the initial bank energy goes into heating the gas and 10% into the magnetic field at the first maximum contraction; for $dI/dt \sim 10^{11}$ amps/sec ($H_z = 0$) 11% goes into the field and only 6.5% into the gas at this time. The energy input to the gas is found to become limited well before peak current, and the initial implosion is not followed by much of the subsequent isentropic compression heating which might be expected, so that the temperature never exceeds about 300 eV; it follows that most of the observed neutron yield ($\sim 10^6$) is not thermonuclear. Data obtained from the electrical waveforms and from high-speed photographs are presented and analysed, and it is shown that large currents build up outside the central plasma column after the first maximum contraction. Further evidence suggests that these currents flow in wall material boiled off by radiation from the central column. These wall effects account for the absence of further isentropic compression heating and also contaminate the central plasma column; they are such as to prevent any significant gain in heating by increasing the power input to the discharge. It is concluded that similar effects probably account for the failure to achieve a thermonuclear temperature in many high power pinch experiments.

1. Introduction

The rapid heating of a plasma is desirable for a number of reasons. For example, if a cold gas is heated up it passes through a stage when it is only partially ionized during which time it loses energy very rapidly by transitions involving bound states*. It is desirable to minimise the duration of this phase by making the power input large compared with the radiated power. Also, the containment problem is simpler at high temperatures. Heating by rapid magnetic compression (see, for example, Andrianov et al. [1], Bickerton [2], Fitch [3]) is one of the most promising methods of obtaining a hot plasma. The simplest geometry for this approach is the linear *z*-pinch.

In earlier experiments on fast linear *z*-pinches carried out at A.W.R.E. the neutron emission was studied at high intial rates of current rise ($dI/dt(0)$); this work showed that even from high-power discharges the neutron emission was not thermonuclear (Bodin et al. [4]). There followed a more detailed investigation of such discharges, with both high and low values of $dI/dt(0)$, (hereafter described as "fast" and "slow"), both with and without an axial magnetic field (H_z). The main qualitative features of fast and slow discharges and a few of the measurements on rapid compression heating, were reported by Bodin et al. [5].

In this paper, which is divided into two main parts, the experiments on rapid compression heating are described in detail. The first part discusses the energy put into the gas and the magnetic field up to the time of the first maximum contraction. It is then shown that in high power discharges the energy input to the gas by compression never significantly exceeds its value at that time. The second part of the paper presents and analyses results which show how the heating is limited by wall effects. The consequences of these effects are discussed.

2. Apparatus and method of measurement

The experiments were carried out using the Maggi condenser bank, described by Fitch and McCormick [6]. The capacity used was 100 μf and most of the measurements were made at 20 kV. The total inductance, L_B, external to the gas discharge was 5×10^{-9} henries for fast discharges and 200×10^{-9} henries for slow discharges, giving $dI/dt(0)$ in the gas of 1.5×10^{12} and 10^{11} amps/sec respectively. A clear quartz discharge tube, 34 cms long, 15 cms bore and 8 mm wall thickness was used, with plane copper electrodes. Weak preionization was always used in the experiments described. Streak photographs were taken and the voltage (V), current (I) and dI/dt were recorded.

3. The electrical characteristics of discharges with high and low initial rates of current rise

Examples of the voltage and current waveforms are shown in figure 1 for fast (A) and slow (B) discharges. A (above) shows a typical current oscillogram for a discharge in which the bank inductance is comparable to the initial tube inductance, but much less than the final tube inductance. In this case the current rises very rapidly to a high value (360 kA in 0.6 μ sec), and then falls to a lower value (208 kA) at the first maximum contraction (first pinch) of the plasma. This form of the variation of current with time is desirable because (see Fitch [3]) at the first maximum contraction more energy can go into the gas than into the magnetic field associated with

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* This has also been pointed out recently by J. L. Tuck in Phys. Rev. Letters, 3, 313 (1959).

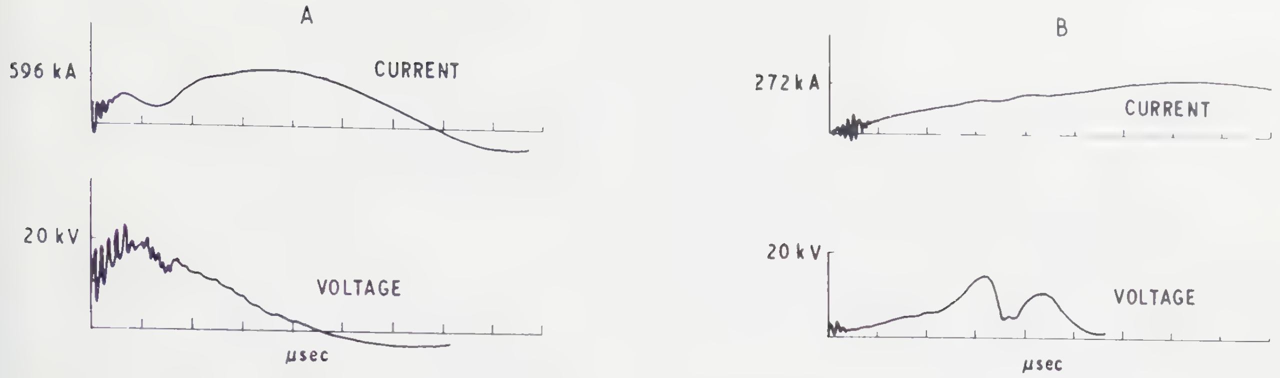

Fig. 1 Tracings of typical current and voltage waveforms at high and low dI/dt (0), $H_z = 0$.
A. dI/dt (0) $\sim 1{,}5 \cdot 10^{12}$ amp/sec, $V_B = 20$ kV, $p_0 = 320\,\mu$ D_2 B. dI/dt (0) $\sim 10^{11}$ amps/sec, $V_B = 21$ kV, $p_0 = 350\,\mu$ D_2.

the discharge. (See section 4.) The voltage is not sensitive to changes in the tube inductance when the bank inductance is small, and the oscillogram in this case (A) does not show any of the rapid fluctuations at the time of the pinch which are characteristic of a high-inductance bank (see B). The high frequency oscillations seen on the voltage oscillogram, and to a lesser extent on the current, are due to cable reflections in the bank (Fitch and McCormick [6]). The current in the slow discharge (B) is predominantly determined by the large series inductance and shows only slight evidence for pinching. In this case the current at the first maximum contraction is 190 kA.

The application of H_z results in several apparently stable oscillations of the plasma, as found by Bezbatchenko et al. [7] The oscillations were seen on streak photographs and also on the current waveform for a low bank inductance (Fig. 2). The peak current is larger for $H_z = 0$, contrary to predictions based on simple inductance considerations. (See also section 6.)

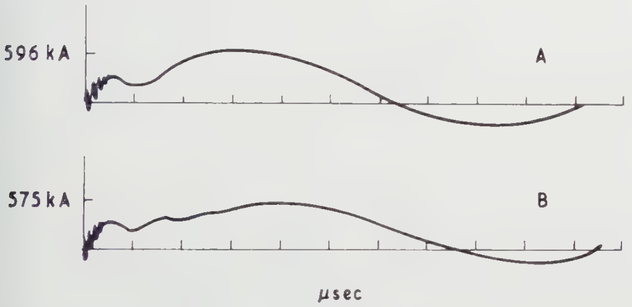

Fig. 2 Tracings of typical current waveforms with and without H_z. $p_0 = 350\,\mu$ D_2, $V_B = 20$ kV, dI/dt (0) $\sim 1.5 \cdot 10^{12}$ amps/sec.
A. $H_z = 0$ B. $H_z = 1480$ oe

4. The particle and magnetic energy in discharges with high and low initial rates of current rise

THE METHOD OF CALCULATING THE DISCHARGE ENERGY

The total amount of electrical energy put into the discharge tube up to time, t, is

$$\int_0^t V I \, dt \tag{1}$$

where V is the voltage across the tube and I the current. Using equation (1) it can be shown* that at time t

$$\int_0^t V I \, dt = \tfrac{1}{2} L I^2 + \tfrac{1}{2} \int_0^t I^2 (dL/dt)\, dt \tag{2}$$

where L = tube inductance.

By definition $\tfrac{1}{2} LI^2$ is the energy in the azimuthal magnetic field, and therefore the last term must represent the total mechanical work done on the system as a result of the motion (due to the magnetic forces) of current-carrying conducting material, i. e. as a result of changes of inductance. The total mechanical work done on the discharge is

$$W = \tfrac{1}{2} \int I^2 (dL/dt)\, dt \tag{3}$$

Equation (3) is independent of the geometry since W includes the work done due to the motion of all the currents in the tube. Where all the currents flow in a central plasma column all of the mechanical work is done on this column.

For discharges with no axial magnetic field W goes into particle energy W_g and

$$W_g = W \tag{4}$$

For discharges with H_z

$$W_g = W - W' \tag{5}$$

where W' is the energy required to compress the trapped field.

The total magnetic energy, W_H, in the discharge is, for unstabilized discharges,

$$W_H = W_{H\theta} = \tfrac{1}{2} L I^2 \tag{6}$$

The total magnetic energy in the discharge for discharges with H_z is

$$W_H = W_{H\theta} + W_{Hz} \tag{7}$$

where W_{Hz} is the energy in the trapped axial field.

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* Resistive heating, which can be shown to be small in times of interest, is neglected.

THE PARTICLE AND MAGNETIC ENERGY IN THE DISCHARGE UP TO THE TIME OF THE FIRST MAXIMUM CONTRACTION

Discharges without axial magnetic field

Figure 3 shows the particle and magnetic energies in the discharge as functions of radius up to the first maximum contraction for the fast and the slow discharges. In the fast discharges it is found that whereas W_g rises steadily to a maximum at the first maximum contraction, W_H levels off so that $W_g/W_H = 1.5$ at this time, i.e. more energy is put into the gas than into the magnetic field. At the first maximum contraction about 15% of the total energy initially in the bank (20 kJ) has gone into the gas, and about 10% into the magnetic field.

In the slow discharge W_H is always greater than W_g and both increase steadily as the compression proceeds. At the first maximum contraction 6.5% of the energy in the bank (20 kJ) has gone into the gas and 12% into the magnetic field. In the slow discharge most of the work is done at small radii, in contrast to the fast discharge in which a substantial amount of energy is put into the gas during the early part of the compression, when the current (Fig. 1) is high.

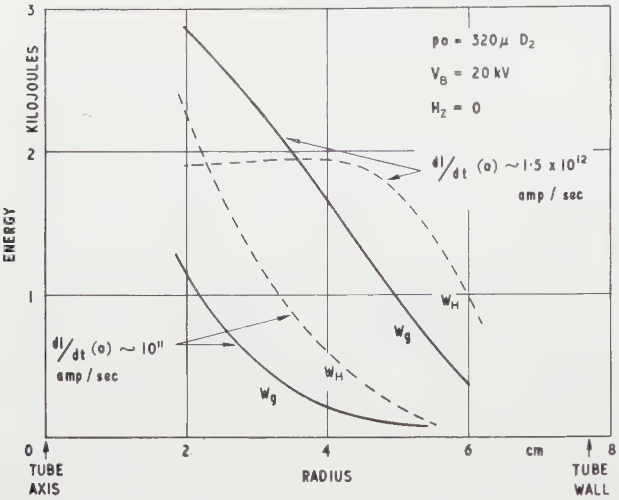

Fig. 3 Total particle energy (W_g) and magnetic energy (W_H) in the discharge as functions of radius at high and low dI/dt (0) without axial magnetic field.

There is a gain by a factor 7 in mean power input and by a factor 2.3 in W_g for the fast discharge. An increase in W_g by a factor of only 2.3 for a factor 15 increase in the inital dI/dt is not unexpected when dI/dt (0) is changed by varying the bank inductance, using a relatively low voltage. It arises because the very high initial dI/dt in the low inductance case is only sustained for a small fraction of the compression time near the start while the tube inductance remains comparable to the bank inductance.

At the first maximum contraction the plasma radius in the fast discharge is larger by a factor 2.1 than that for the slow discharge; this is a consequence of the fact that more energy is put into the gas in the former case during the earlier part of the compression, and the plasma pressure limits the contraction at a larger radius. Another example of this is seen in the streak photographs in fig. 9, for $H_z = 0$, which show that the plasma radius at the first maximum contraction increases with the voltage.

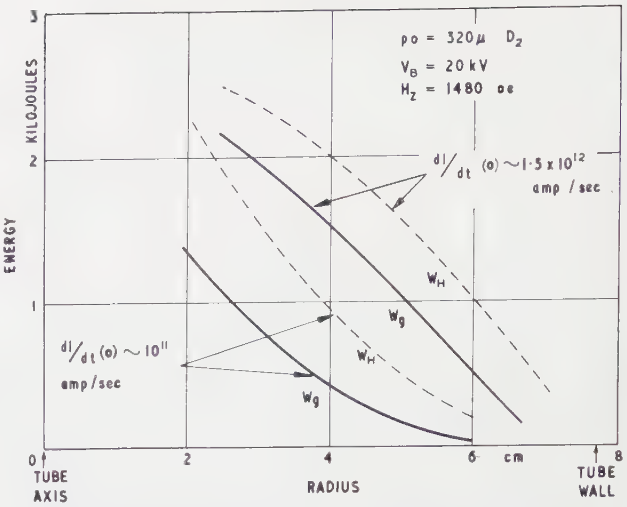

Fig. 4 Total particle energy (W_g) and magnetic energy (W_H) in the discharge as functions of radius at low and high dI/dt (0) with axial magnetic field, H_z.

Discharge with axial magnetic field

Moderate axial magnetic fields were applied to reduce the fastest growing modes and improved stability was found (Bodin et all. [5]) for fields ~ 1000 oe. Fig. 4 shows the particle and magnetic energy (including that in the trapped field) in the discharge as functions of radius up to the first maximum contraction for the fast and the slow discharge with $H_z = 1480$ oe. It is found that with an axial magnetic field W_g is always less than W_H in both cases, although only by 13% for the fast discharge. Comparing figures 3 and 4 it is seen that the application of H_z does not have a marked effect on W_g and most of the magnetic energy is still in the H_θ field.

Ion temperatures

The values of W_g and ion temperature, T_i, at the first maximum contraction are shown as functions of initial pressure for the fast discharge in fig. 5. For $H_z = 0$ the concept of temperature is fully valid in most cases and is always approximately valid, as ion thermalization will

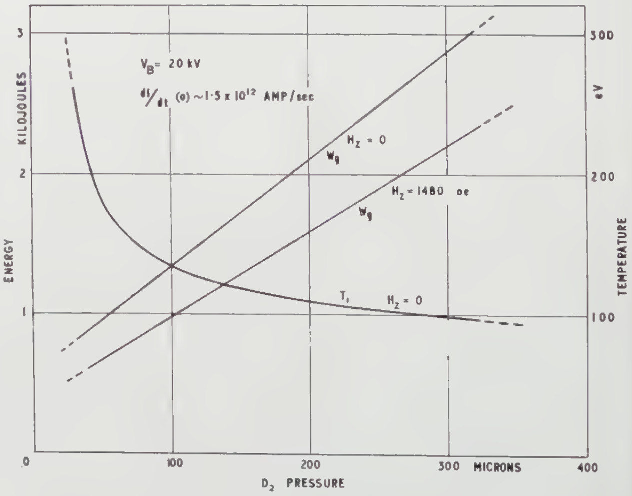

Fig. 5 Total particle energy (W_g) in the discharge and ion temperature (T_i) as functions of initial pressure at high dI/dt (0) with and without axial magnetic field, H_z.

occur in $\sim 0.2\,\mu$sec in the longest case given ($T_i \sim 300$ eV at intitial pressure $\sim 25\,\mu$). For $H_z = 1480$ oe the trapped field will delay the thermalization unless, for example, there is a shock wave, and values of T_i are not given for this case. T_i is calculated from W_g assuming no losses and that all the energy is in the ions, i.e.

$$T_i = \frac{2}{3} W_g / Nl$$

where Nl = number of deuterons in the tube.

It is seen that the highest temperature which could have been achieved at the first maximum contraction is 300 eV at 25 μ, the lowest pressure at which a discharge can be initiated with a bank voltage, V_B, of 20 kV.

THE WORK DONE ON THE GAS UP TO PEAK CURRENT-MAXIMUM TEMPERATURES ACHIEVED

Figs. 6 and 7 show W_g and W_H as functions of time for $H_z = 0$ and 1480 oe respectively*. In the fast discharge W_g rises to a maximum at the time of the first maximum contraction, and then falls as the plasma expands against the field; as the plasma contracts for a second time W_g begins to rise again but it is found that, for $H_z = 0$, it never exceeds the value reached at the first maximum contration. For $H_z = 1480$ oe (see fig. 7) it only exceeds the early value by 20% during the later stages. Thus, for $H_z = 0$ there is no further compression heating after the first maximum contraction, while for $H_z = 1480$ oe there is only a little subsequent heating. The observation that the plasma temperature never significantly exceeds that achieved at the first maximum contraction is general in the present experiments and is discussed in the next section.

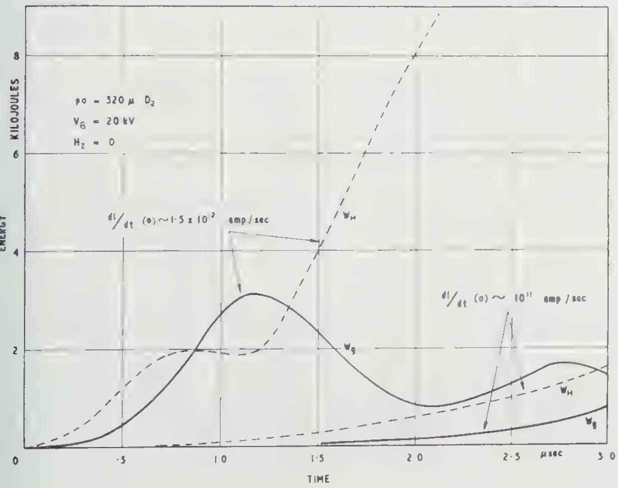

Fig. 6 Total particle energy (W_g) and total magnetic energy (W_H) in the discharge as functions of time at low and high dI/dt (0), $H_z = 0$.

The thermonuclear yield expected from the calculated ion temperature in these experiments (see preceding section) is only 10^3—10^4, which confirms the conclusions reached in the study of the neutron emission (Bodin et al.[4]) that most of the observed yield ($\sim 10^5$) was not thermonuclear.

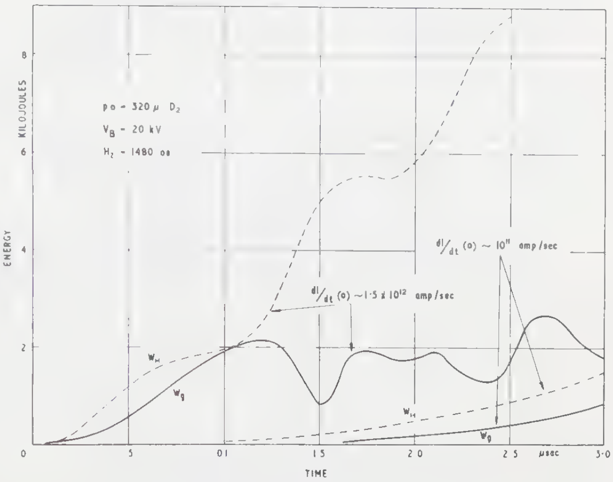

Fig. 7 Total particle energy (W_g) and total magnetic energy (W_H) in the discharge as functions of time at low and high dI/dt (0) with axial magnetic field.

5. The heating limitation in high-power Z-pinches

The processes limiting the heating were revealed when data from the streak photographs and the electrical waveforms were compared.

THE METHOD OF ANALYSIS OF THE STREAK PHOTOGRAPHS AND THE ELECTRICAL MEASUREMENTS

The inductance $L(t)$ of the discharge can be obtained from V and I by the equation

$$V = (d/dt)(LI) \tag{8}$$

which can be written

$$L(t) = \frac{\int_0^t V\,dt}{I(t)} \tag{9}$$

Resistive terms have again been neglected because, except in the early sheath formation stages and in the later stages when impurities from the walls enter the discharge, the current and voltage waveforms are consistent with a purely inductive impedance of the discharge. Furthermore, if resistance is included the main conclusion (that I_e can substantially exceed I_s — see below) would be strengthened.

The plasma radius $r(t)$ can be found from the inductance using the relation

$$L(t) = 2l \log[R/r(t)] \times 10^{-9} \text{ henries} \tag{10}$$

where R = radius of outer coaxial current return and l = tube length, on the assumption that the current flows in a thin sheath. (The inductance for a thick sheath or for a uniform current is larger.) The radius of the current sheath found in this way from the electrical measurements is denoted by r_e. The radius of the outer boundary of the central luminous column (r_s) is determined directly from the streak photographs. Values of the current corresponding to r_e and r_s are, respectively, I_e (measured directly by the monitor) and I_s (determined from the streak measurements, using equation (10) to find a value of $L(t)$ from r_s and inserting this value and the observed voltage into equation (9)). I_s only equals I_e if all the current flows on the outside of the luminous central column; in this case r_s and r_e are also equal.

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* W_g after the first maximum contraction includes a small contribution due to currents which are found (see section 5) to grow outside the central plasma column after this time. At earlier times all W_g goes into the central plasma column.

Results obtained from analysis of the streak and electrical measurements

Fig. 8 (A, above) shows I_e, I_s and also the current given by theory as functions of time for three different discharges; fig. 8 (B, below) shows the corresponding values of r_e, r_s and theory. The initial gas pressure is arranged so that in all three cases the mass of gas, M, per cm length of the tube is approximately the same, the voltage is 20 kV in each case so the pinch-time scaling factor $(V^2/M)^{\frac{1}{4}}$ is the same for the three discharges to a good approximation. The theoretical curves were computed for the present experiments by Taylor and Nisbet [8] from the Rosenbluth [9] snowplough equations, taking into account the effect of condenser-bank inductance and H_z.

It is found that I_e, I_s and theory are in fairly good agreement in the early part of the discharge in all cases (see fig. 8, A). The biggest discrepancy during this time, for deuterium with $H_z = 0$, can be explained by uncertainties in determining r_s during the very early stages when the whole curve is very sensitive to the values of r_s just after the plasma leaves the walls.

The agreement between I_s and I_e in the early stages is sustained for more than two successive contractions in the case of deuterium with $H_z = 1480$ oe. For deuterium with $H_z = 0$, I_e begins to exceed I_s soon after the first maximum contraction; I_s falls as the central column contracts for a second time (see observed radius time curve) but I_e continues to rise. For nitrogen, I_e becomes much greater than I_s before the first maximum contraction and the first dip in the observed current occurs well before the first maximum contraction of the central column. When the discrepancy appears the tube is presenting a lower inductance to the bank than it would if all the current flowed in the central region. This lower inductance represents a larger "effective" radius for the current channel and the radius/time curves bear this out. For nitrogen r_e is about 5 cm while r_s is less than 0.5 cm at the first maximum contraction, and most of the current must flow in the outer regions of the tube. The observed peak current for a nitrogen discharge is 800 kA, while for deuterium it is 600 kA, although the radius of the luminous central column is smaller for nitrogen.

The evidence from the current measurements could be consistent either with all the current flowing at radii in the neighbourhood of r_e, or with a current I_s flowing on the outside of the central column together with other

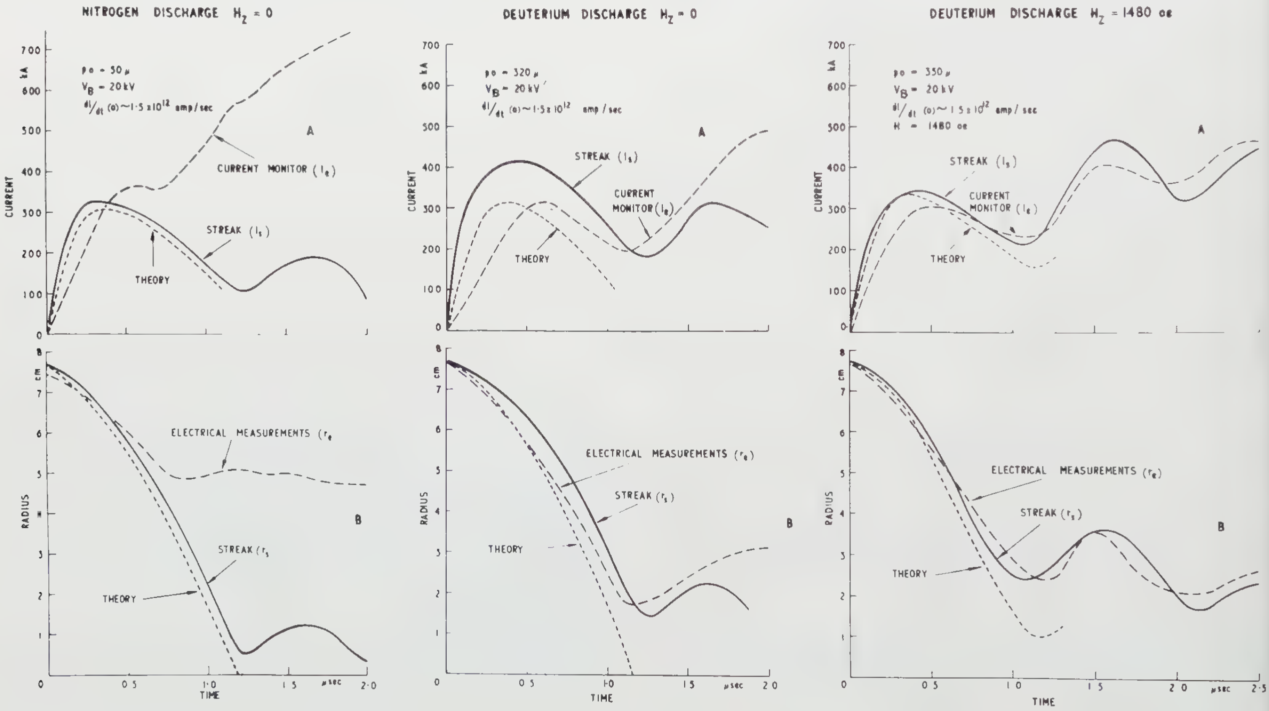

Fig. 8 A (above) Comparison between the current/time variation measured directly (I_e), and calculated from the streak measurements (I_s), and according to theory for three different discharges.

B (below) Comparison between the radius/time variation determined from the streak photographs (r_s), calculated from the observed current (r_e), and according to theory for the same three discharges.

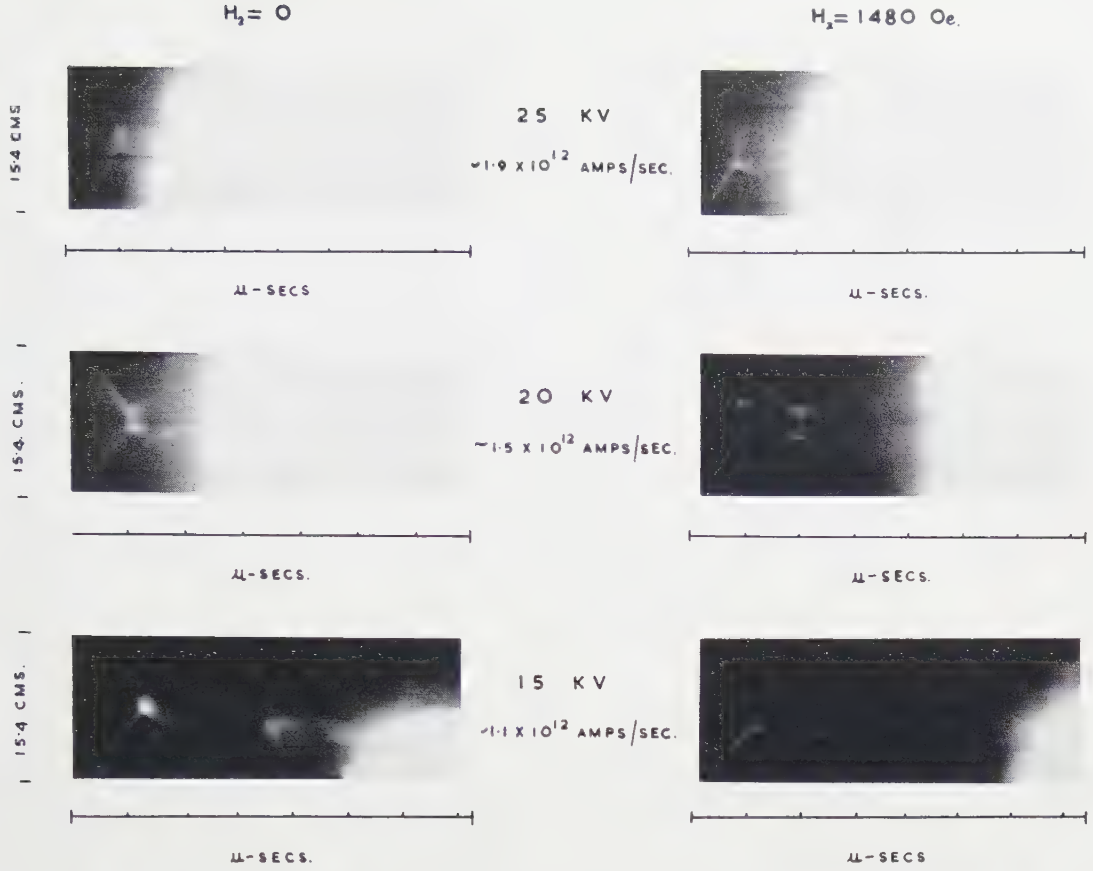

Fig. 9 Radial streak photographs at different values of bank voltage with and without H_z ($p \sim 300$—$350\ \mu$). Diameter of tube $(2R) = 15.4$ cm. r_s is measured from the outer edge of the luminous sheath.

currents at radii $> r_s$. However, the agreement between I_s and theory and r_s and theory during the whole of the implosion phase in all three cases shows that the dynamical behaviour of the discharge is consistent with a current I_s flowing in the central column, even though, for the nitrogen discharge, the total current becomes much greater than I_s before the first maximum contraction. After the first maximum contraction of the radius/time behaviour of the central column is still largely consistent with a current I_s flowing in it, and the most likely explanation of the discrepancy is the growth of additional current outside the luminous central column.

The observation that I_e can exceed I_s is a general one, of which the cases shown are typical. The current discrepancy is observed to depend on the energy input to the gas, the compression ratio and the presence of high-Z impurities in the gas. The growth of current outside the central region has also been observed by Andrianov et al. [1].

Qualitative observations from the streak photographs

At about the time current begins to grow outside the central volume an increase in the background luminosity is observed on streak photographs, consistent with a marked increase in luminosity near the discharge tube walls. Similar effects have been observed by Burganov et al. [10] and Komelkov [11]. The onset of this light varies with energy input, compression ratio and the presence of impurities in the gas in the same way as the current discrepancy. In nitrogen discharges faint background light appears during the initial implosion and becomes very intense at the first maximum contraction. In deuterium discharges at pressures $> 100\ \mu$ this light appears at or soon after the first maximum contraction and is very much brighter than the central colum, which becomes completely indistinguishable. Figure 9 shows streak photographs in deuterium at different bank voltages (V_B) both with and without an axial magnetic field. The middle two pictures correspond to the curves for deuterium in fig. 8 and it is seen that with $H_z = 0$ the onset of background light and the current discrepancy both begin at 1.5 μsec; with $H_z = 1480$ oe the light comes later, and I_s and I_e are in agreement for much longer.

At initial deuterium pressure $\lesssim 100\ \mu$ the central column is invisible, but background light still appears very brightly at or soon after the first maximum contraction (determined electrically). Even in discharges with 99.99% pure deuterium the same general phenomena are observed on the streak photographs. The addition of $2^1/_2$% of nitrogen to a deuterium discharge radically alters its character, and it behaves more like a pure nitrogen discharge than a deuterium discharge in the respects described above.

Interpretation of the results obtained from the streak photographs and electrical measurements

The observations decribed above can largely be explained by the following qualitative theory. At some time during the discharge the surface of the tube wall is vapourised by radiation from the central plasma, and current (in the same direction as the central current) builds up in the gas produced in this way, which becomes highly luminous due to the presence of oxygen and silicon. This model is consistent with the observation that wall effects are more pronounced and occur sooner if the energy input is raised, or if the content of impurities is increased or if the compression ratio becomes larger.

A discharge in wall material, once initiated, will grow rapidly and the radiation need only provide a trigger. As the temperature of the wall surface depends on the radiation flux, the total intensity of radiation required can be quite small. It is easy to account for large radiation fluxes in heavy gases, but in deuterium the bremsstrahlung can be shown to be negligible and the radiation could either come from impurities drawn in from the walls in the initial stages or from partially ionized deuterium, or possibly from some kind of turbulence. It has been shown (see, for example, McWhirter [12] and Kogan [13]) that the radiation from a deuterium plasma can be greatly increased by the addition of very small amounts of impurity. The wavelength of the radiation must be short, as wall effects have been observed when no visible light is detected from the central region; furthermore, quartz is transparent to wavelengths >2000 Å.

Wall effects can also be initiated by plasma which escapes from the central region and hits the walls. In some slower unstabilized discharges in light gases there is evidence for plasma jetting out and causing a highly localised increase in luminosity at the walls (see, for example, fig. 9 for $V_B = 15$ kV, $H_z = 0$). For $dI/dt = 1.5 \times 10^{12}$ amps per sec the onset of wall light is usually symmetrical (see fig. 9 for $V_B = 20$ and 25 kV and $H_z = 0$ and 1480 oe), and occurs in times rather short to be explained by plasma hitting the walls; in these conditions the radiation mechanism is much more likely.

Further evidence for a discharge in wall material is the appearance of a second, highly luminous, sheath which contracts slowly in some conditions at low power; this sheath begins to form when the background of light from the vicinity of the walls appears. In other conditions the luminous region appearing near the walls is seen to move in slowly without forming a distinct sheath. The electron temperature in the discharge in vapourised wall material will be low and resistive effects important, leading to a thick current sheath.

These wall effects can explain many anomalies; for example, the appearance of the first "dip" in the current waveform well before the first maximum contraction in nitrogen discharges; the observation that the peak current for discharges in nitrogen is 30% greater than for those in deuterium; the decrease in peak current for deuterium discharges when H_z is applied.

The influence of wall effects on the compression heating

The production of conducting material at the walls and the subsequent growth of currents in this material screens off the inductive electric field from the central column and the rate of current rise in this column falls. The current in the central plasma is therefore limited and there is little further compression heating. At later times the central plasma becomes heavily contaminated, as observed by Andrianov et al. [1], Komelkov [11] and others.

If attempts are made to increase the energy input to the gas by, for example, increasing the voltage, then wall effects appear earlier, and the heating becomes limited sooner, with negligible gain. Fig. 9 shows evidence for this from streak photographs at $V_B = 15$, 20 and 25 kV; it is seen that the background light comes sooner at higher voltages. Increasing the axial magnetic field or reducing the voltage both reduce the energy density in the plasma at the time of the first maximum contraction and are observed to delay the onset of wall effects. In fact it is found that over a wide range of conditions there is a limit to maximum energy input to the gas. There is evidence for this in fig. 6 and 7 which show that the maximum value of W_g is almost the same with or without H_z. With $H_z = 0$ more energy is put into the gas at the first maximum contraction, and there is no further heating. With H_z, the energy input to the gas is less at the first maximum contraction, wall effects are delayed, and there is some further heating—sufficient to increase W_g to about the value achieved with $H_z = 0$. In general the results suggest that, in the present experiments, as soon as the energy density in the central plasma exceeds a certain value, wall effects appear and prevent further energy input to the central column. In some experiments using very high voltages (120 kV) Andrianov et al. [14] found that large currents flowed at the walls right from the start of the discharge.

6. Conclusion

In low-power z-pinches insufficient energy is put into the discharge to achieve a thermonuclear temperature, even if the initial implosion is followed by further compression heating; in this case such compression takes so long that the heating is limited by instabilities, impurities from the walls and other cooling processes. In very fast pinches wall effects limit the heating at a much earlier stage and so in no case is a thermonuclear plasma produced. However, the ratio of particle to magnetic energy in the high-power case, before the heating becomes limited, is very favourable for further compression if wall effects can be eliminated. It seems probable that the type of effects observed in the present experiment are responsible for the failure to achieve a thermonuclear temperature in many of the high-power, fast-pinch experiments so far carried out.

7. Acknowledgements

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ENERGY DISTRIBUTION OF NEUTRONS PRODUCED BY A THERMONUCLEAR REACTION*

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The energy distribution of neutrons in a thermonuclear reaction is calculated. It is assumed that the reacting mixture of gases is in thermodynamic equilibrium and that the differential cross section for neutron production is isotropic in the center of mass system.

The energy distribution of neutrons produced by a thermonuclear reaction has recently become of interest because of the possibility of producing such reactions [1] in the laboratory. Some calculations [2] have been made previously but were completely different in approach from the methods presented herein.

In order to calculate the neutron energy distribution, it is assumed that the reacting mixture of gases is in thermodynamic equilibrium at a temperature T and that the equilibrium is not disturbed appreciably by the nuclear reactions which are taking place. It is also assumed that the differential cross section for neutron production is isotropic in the center of mass coordinate system of two interacting particles.

It follows [3] from the assumptions made above that the number of pairs of ions with velocities in the range $d\mathbf{c}_1$, $d\mathbf{c}_2$ is

$$\frac{\nu_1 \nu_2}{\gamma_{12}} \frac{(m_1 m_2)^{3/2}}{(2\pi k T)^3} \exp\left\{-\frac{1}{2kT}(m_1 c_1^2 + m_2 c_2^2)\right\} d\mathbf{c}_1 d\mathbf{c}_2 \quad (1)$$

where k = Boltzmann's constant
T = temperature
$\mathbf{c}_1$, $\mathbf{c}_2$ represents the velocities of ions of mass m_1 and m_2 respectively
ν_1, ν_2 = numbers of ions per unit volume of mass m_1, m_2 respectively

$$\gamma_{12} = \begin{cases} 1 \text{ if } m_1 + m_2 \\ 2 \text{ if } m_1 = m_2 \end{cases}$$

The differential rate for neutron emission into the solid angle $d\Omega = 2\pi \sin\theta \; d\theta$ is obtained from (1) by multiplying by $g\sigma(g)$ where $g = |\mathbf{c}_1 - \mathbf{c}_2|$ and $\sigma(g)$ is the appropriate reaction cross section, namely

$$\frac{\nu_1 \nu_2}{\gamma_{12}} \frac{(m_1 m_2)^{3/2}}{(2\pi k T)^3} g\sigma(g) \exp\left\{-\frac{1}{2kT}(m_1 c_1^2 + m_2 c_2^2)\right\} d\mathbf{c}_1 d\mathbf{c}_2 \frac{d\Omega}{4\pi} \quad (2)$$

The energy distribution of neutrons observed in a coordinate system at rest relative to the gas may be obtained from (2) by transformation of coordinates and subsequent integration over variables extraneous to the problem. The transformation involves a change of variables from $\mathbf{c}_1$, $\mathbf{c}_2$ and $\cos\theta$ to a new set $\mathbf{g}$, $\mathbf{G}$ and E the neutron energy. It is convenient to perform the transformation in two separate steps; the first [4]

$$\begin{aligned} \mathbf{c}_1 &= \mathbf{G}' + \frac{m_2}{m_0}\mathbf{g}' \\ \mathbf{c}_2 &= \mathbf{G}' - \frac{m_1}{m_0}\mathbf{g}' \\ \cos\theta &= \cos\theta' \end{aligned} \quad (3)$$

where $m_0 = m_1 + m_2$

The second transformation is

$$\begin{aligned} \mathbf{g}' &= \mathbf{g} \\ \mathbf{G}' &= \mathbf{G} \\ \cos\theta' &= \frac{\frac{2}{m}E - (g)^2 - u^2}{2Gu} \end{aligned} \quad (4)$$

In the last equation of (4) E is the neutron kinetic energy measured in the laboratory coordinate system while u is the neutron velocity in the center of mass coordinate system and m is the neutron mass. The Jacobian transforming the original set of differentials to the new set $d\mathbf{g}$, $d\mathbf{G}$, dE is

$$\frac{\partial(\mathbf{c}_1, \mathbf{c}_2 \cos\theta)}{\partial(\mathbf{g}, \mathbf{G}, E)} = \frac{\partial(\mathbf{c}_1, \mathbf{c}_2 \cos\theta)}{\partial(\mathbf{g}', \mathbf{G}', \cos\theta')} \frac{\partial(\mathbf{g}', \mathbf{G}', \cos\theta')}{\partial(\mathbf{g}, \mathbf{G}, E)} \quad (5)$$

Chapman and Cowling [5] have evaluated the first determinant on the right hand side of (5) and show it to bequal to unity, while the second determinant is

$$\frac{\partial(\mathbf{g}', \mathbf{G}', \cos\theta')}{\partial(\mathbf{g}, \mathbf{G}, E)} = \frac{1}{muG} \quad (6)$$

where $G = |\mathbf{G}|$.

Limits of domaine for g, G and E are obtained from the original limits of the old variables subject to the restrictions implied by the transformations, thus it is found that

$$\begin{gathered} 0 < g < \infty \\ \sqrt{\frac{2E}{m}} - u < G < \sqrt{\frac{2E}{m}} + u \\ 0 < \mathrm{E} < \infty \end{gathered} \quad (7)$$

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Since the limits of G are determined by E and u, the G integration must be performed first. By utilizing the Jacobians found above, the new differential distribution can be written,

$$\frac{\nu_1\nu_2}{\gamma_{12}}\frac{(m_1 m_2)^{3/2}}{(2\pi kT)^3} g\sigma(g)$$
$$\exp\left\{-\frac{1}{2kT}\left(m_0 G^2+\frac{m_1 m_2}{m_0}g^2\right)\right\} d\mathbf{G}\, d\mathbf{g}\frac{dE}{2muG} \qquad (8)$$

Integration of (8) over all directions of G and $\mathbf{g}$ yields

$$\frac{\nu_1\nu_2}{\gamma_{12}\pi}\frac{(m_1 m_2)^{3/2}}{(kT)^3} g\sigma(g)\exp\left\{-\frac{1}{2kT}\left(m_0 G^2+\frac{m_1 m_2}{m_0}g^2\right)\right\}$$
$$\frac{G^2\, dG\, g^2\, dg\, dE}{m u G} \qquad (9)$$

Next integration with respect to $\mathbf{G}$ and g in that order yields the desired energy distribution of neutrons, namely

$$\frac{\nu_1\nu_2(m_1 m_2)^{3/2}\, dE}{\gamma_{12}\pi(kT)^2 m_0 m}\int_0^\infty\frac{dg\, g^3\sigma(g)}{u}\exp\left\{\frac{-m_1 m_2}{2m_0 kT}g^2\right\}$$
$$\left\{\exp\left[-\frac{m_0}{2kT}\left(\sqrt{\frac{2E}{m}}-u\right)^2\right]\right.$$
$$\left.-\exp\left[-\frac{m_0}{2kT}\left(\sqrt{\frac{2E}{m}}+u\right)^2\right]\right\} \qquad (10)$$

For convenience the integration variables will be changed to $E_r = (m_1 m_2/2\, m_0)\, g^2$, and with the substitution

$u^2 = \left[\frac{2}{m}\alpha(Q+E_r)\right]$ where $\alpha = \left(1-\frac{m}{m_0}\right)$, equation (10) becomes,

$$\frac{\nu_1\nu_2(2\, m_0)\, dE}{\pi\gamma_{12}(kT)^2\sqrt{2\, m m_1 m_2\alpha Q}}\int_0^\infty\frac{dE_r\,\sigma(E_r)\, E_r}{\sqrt{1+E_r/Q}}\exp\left\{\frac{-E_r}{kT}\right\}$$
$$\left\{\exp\frac{-m_0}{mkT}\left[\sqrt{E}-\sqrt{\alpha(Q+E_r)}\right]^2\right.$$
$$\left.-\exp\frac{-m_0}{mkT}\left[\sqrt{E}+\sqrt{\alpha(Q+E_r)}\right]^2\right\} \qquad (11)$$

An excellent approximation to the actual form of the neutron energy distribution function (11) may be obtained by noting that $E \cong \alpha Q$ while $E_r \cong kT$ is very much less than E. Under these condition equations (11) can be approximated as

$$\frac{v_1 v_2\, 2\, m_0\, dE}{\pi(kT)^2\gamma_{12}\sqrt{2\, m m_1 m_2\alpha Q}}\exp\left\{\frac{-m_0}{mkT}\left[\sqrt{E}-\sqrt{\alpha(Q+E_r)}\right]^2\right\}$$
$$\int_0^\infty E_r\, dE_r\,\sigma(E_r)\exp(-E_r/kT) \qquad (12)$$

It follows from equation (12) that the width of the neutron energy distribution at e^{-1} of its maximum value is (to terms of first order)

$$\delta E = \sqrt{\frac{mkT}{m_0}}\sqrt{Q} \qquad (13)$$

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ABSTRACTS IN ENGLISH

Plasma oscillations, (I), I. B. Bernstein *(Project Matterhorn, Princeton University, N. J., U.S.A.)* and S. K. Trehan *(New Delhi, India)* Nuclear Fusion **1**, 3 — 41 (1960)

This article is the first part of a review of the theory of wave phenomena in plasmas. The basic kinetic theory is developed, taking into account coulomb collisions. From this, magnetohydrodynamic equations are derived. It is demonstrated that it is legitimate to deal with a closed subset of these equations either in the limit of collision-dominated phenomena, or in the limit where the effective phase velocities of the phenomena of interest are much greater than thermal speeds. In case of collision-dominated plasmas the theory of transport coefficients is discussed.

These equations are then applied to an extensive treatment of small-amplitude wave phenomena in plasmas. A discussion of the dissipative effects on hydromagnetic waves is given. Hydromagnetic waves are also considered from the Chew, Goldberger and Low theory. Longitudinal and transverse oscillations in current-carrying plasmas are also discussed.

Oscillations of a cylindrical plasma are considered and the phenomenon of ion cyclotron resonance is discussed. The possibility of radiation by plasma oscillations by a uniform sphere is exhibited. Some general results on the stability of longitudinal electron oscillations in non-uniform plasmas are given. A brief treatment of large amplitude electron oscillations is given and the breaking of these oscillations as a dissipative mechanism for the organized plasma motion is discussed.

Part II of this paper, to appear in a future issue of this journal, will be devoted to the discussion of plasma oscillations directly from the kinetic theory.

Rayleigh-Taylor instabilities of magnetically accelerated plasma, T. S. Green and G. B. F. Niblett *(Atomic Weapons Research Establishment, Aldermaston, Berks., England)* Nuclear Fusion **1**, 42 — 46 (1960)

High speed photography has been used to study the compression of a deuterium plasma by an axial magnetic field generated by a wide single-turn coil. In this experiment the inductance of the coil is small compared with that of the external condenser bank so that the plasma implodes at a constant acceleration of $5 \cdot 10^{12}$ oms/sec^2. Flutes which develop on the outer surface of the plasma are interpreted as magnetohydrodynamic Rayleigh-Taylor instabilities produced by the inward accelaration of the interface separating magnetic field and plasma. The observed growth rates are in agreement with simple theory. It is shown that the ratio of the diffusivities of magnetic field and momentum determines whether the plasma resistivity or its viscosity is the dominant mechanism for damping the short wavelength instabilities. At the temperatures and densities of this experiment the resistivity is primarily responsible for the damping and the wavelength of "maximum instability" calculated on this basis agrees with the observed value.

A necessary condition for hydromagnetic stability of plasma with axial symmetry, C. Mercier *(Group de recherches de l'Association Eurotom-C.E.A., Fontenay-aux-Roses, Seine, France)* Nuclear Fusion **1**, 47 — 53 (1960)

In this work a necessary condition is derived for the stability of a plasma with axial symmetry. This condition corresponds to displacements which are localized in the neighborhood of those surfaces of constant pressure on which the lines of force are closed. This criterion reduces to Suydam's criterion in the case of cylindrical symmetry.

Rapid compression heating of a plasma in the linear, z-pinch H. A. B. Bodin, A. A. Newton and N. J. Peacock *(Atomic Weapons Research Establishment, Aldermaston, Berks., England)* Nuclear Fusion **1**, 54 — 61 (1960)

The electrical characteristics of discharges with high and low initial rates of current rise are briefly described and the electrica measurements are used to determine the work done on the plasma by rapid compression. The particle and the magnetic energy in the discharge are discussed for both high and low initial rates of current rise ($dI/dt(0)$) and for discharges with and without an axial magnetic field (H_z). It is shown that the gas is heated much more efficiently both with and without H_z at high $dI/dt(0)$. For $dI/dt(0) \sim 1.5 \cdot 10^{12}$ amps/sec ($H_z = 0$) 15% of the initial condenser bank energy goes into heating the gas and 10% into the magnetic field at the first maximum contraction; for $dI/dt \sim 10^{11}$ amps/sec ($H_z = 0$) 11% goes into field and only 6.5% into the gas at this time. The energy input to the gas is found to become limited well before peak current, and the initial implosion is not followed by much of the subsequent isentropic compression heating which might be expected, so that the temperature never exceeds about 300 eV; it follows that most of the observed neutron yield ($\sim 10^6$) is not thermonuclear. Data obtained from the electrical waveforms and from high-speed photographs are presented and analysed, and it is shown that large currents build up outside the central plasma column after the first maximum contraction. Further evidence suggests that these currents flow in wall material boiled off by radiation from the central column. These wall effects account for the absence of further isentropic compression heating and also contaminate the central plasma column; they are such as to prevent any significant gain in heating by increasing the power input to the discharge. It is concluded that similar effects probably account for the failure to achieve a thermonuclear temperature in many high power pinch experiments.

Energy distribution of neutrons produced by a thermonuclear reaction, W. R. Faust and E. G. Harris *(U. S. Naval Research Laboratory, Washington, D.C., U.S.A.)* Nuclear Fusion **1**, 62 — 63 (1960)

The energy distribution of neutrons in a thermonuclear reaction is calculated. It is assumed that the reacting mixture of gases is in thermodynamic equilibrium and that the differential cross-section for neutron production is isotropic in the center of mass system.

RESUMÉS EN FRANÇAIS

Oscillations du plasma (I), I. B. Bernstein et S. K. Trehan *(Project Matterhorn, Princeton University, Princeton, N. J., États-Unis d'Amerique)* Fusion Nucléaire **1**, 3 — 41 (1960)

Cet article est la première partie d'une étude d'ensemble portant sur la théorie des phénomènes ondulatoires dans les plasma. Les auteurs y développent la théorie cinétique fondamentale, en tenant compte des chocs coulombiens et parviennent ainsi à établir des équations magnétohydrodynamiques. Ils démontrent que l'emploi d'un sous-ensemble complet de ces équations constitue une approximation légitime pour l'étude de phénomènes dans lesquels soit les chocs prédominent, soit les vitesses effectives de phases sont très supérieures aux vitesses d'agitation thermique. Ils examinent la théorie des coefficients de transport dans le cas de plasmas où les chocs prédominent.

Ces équations sont ensuite appliquées à une étude approfondie des phénomènes ondulatoires de faible amplitude dans les plasmas. Les effects de dissipation sur les ondes hydromagnétiques font l'objet d'un examen critique. Les ondes hydromagnétiques sont également étudiées du point de vue de la théorie de Chew, Goldberger et Low. On examine aussi les oscillations longitudinales et transversales dans les plasma traversés par un courant.

Les auteurs étudient les oscillations d'un plasma cylindrique ainsi que le phénomène de résonance cyclotronique des ions. Ils montrent que les oscillations d'une sphère uniforme du plasma peuvent produire un rayonnement. Ils donnent quelques résultats généraux concernant la stabilité des oscillations longitudinales des electrons dans des plasmas non uniformes.

L'article renferme une brève étude des oscillations de grande amplitude des electrons et de l'interruption de ces oscillations en tant que mécanisme de dissipation pour le mouvement du plasma organisé.

La deuxième partie de l'article, qui paraîtra ulterieurement dans un numéro de cette revue, sera consacrée à l'étude des oscillations du plasma, directement fondée sur la théorie cinétique.

Instabilités de Rayleigh-Taylor dans un plasma soumis à une acceleration magnétique, T. S. Green et G. B. E. Niblett *(Atomic Weapons Research Establishment, Aldermaston, Berkshire, Royaume-Uni)* Fusion Nucléaire **1**, 42 — 46 (1960)

Les auteurs ont eu recours à la photographie ultra-rapide pour étudier la compression d'un plasma de deutérium par un champ magnétique axial créé par un enroulement large à spire unique. Dans cette expérience, l'inductance de la bobine est faible, comparée à celle des condensateurs extérieurs, de telle sorte que le plasma «implose» à une accélération constante de 5×10^{12} cm/sec^2. De l'avis des auteurs, les stries qui apparaissent à la surface extérieure du plasma correspondent aux instabilités magnétohydrodynamiques de Rayleigh-Taylor, produites par l'accélération centripète de l'interface qui sépare le champ magnétique et le plasma. La vitesse de croissance observée concorde avec la théorie. Les auteurs démontrent que, le rapport entre la diffusivité du champ magnétique et celle du moment determine si c'est la résistivité du plasma ou sa viscosité qui contribuent surtout à amortir les instabilités de courte longeur d'onde. Dans les conditions de température et de densité de l'expérience, l'amortissement est dû surtout à la résistivité et la longeur d'onde «d'instabilité maximum», calculée à partir de cette hypothèse concorde avec la valeur effectivement observée.

Un critère nécessaire de stabilité hydromagnetique pour un plasma en symétrie de révolution, Claude Mercier *(Groupe de recherches de l'association Euratom-C.E.A., Fontenay-aux-Roses, Seine, France)* Fusion Nucléaire **1**, 47 — 53 (1960)

Pour ce travail, on établit une condition nécessaire de stabilité pour un plasma en symétrie de révolution. Elle correspond à des déplacements localisés au voisinage des surfaces à pression constante sur lesquelles les lignes de force magnétiques sont fermées. Ce critère se réduit à celui de Suydam en symétrie cylindrique.

Chauffage du plasma par compression rapide dans la striction linéaire avec champ magnétique axial, H. A. B. Bodin, A. A. Newton et N. J. Peacock *(Atomic Weapons Research Establishment, Aldermaston, Berkshire, Royaume-Uni)* Fusion Nucléaire **1**, 54 — 61 (1960)

Les auteurs décrivent brièvement les caractéristiques électriques des décharges avec accroissement initial du courant rapide ou lent, les mesures électriques suivant à determiner l'effet de la compression rapide sur le plasma. Ils étudient l'énergie corpusculaire et l'énergie magnétique inhérentes à la décharge pour des accroissements de courant initiaux ($dI/dt(0)$) rapides et lents, et pour des décharges avec et sans champ magnétique axial (H_z). Ils montrent que le chauffage du gaz est beaucoup plus efficace lorsque la valeur de $dI/dt(0)$ est élevée, qu'il y ait ou non un champ magnétique axial. Pour $dI/dt(0) \sim 1{,}5 \times 10^{12}$ A/s ($H_z = 0$), 15% de l'énergie initiale fournie par les condensateurs servent à chauffer le gaz et 10% sont communiqués au champ magnétique au moment de la première contraction maximum; pour $dI/t \sim 10^{11}$ A/s ($H_z = 0$), 11% sont absorbés par le champ magnétique et 6,5% seulement par le gaz, au moment considéré. On constate que la quantité d'énergie fournie au gaz se trouve limitée bien avant que le courant n'ait atteint sa valeur maximum, et que l'échauffement par compression isentropique qui suit l'implosion initiale est beaucoup plus faible qu'on pourrait le prévoir, de sorte que la température n'est jamais supérieure à environ 300 eV; il s'ensuit que la plus grande partie du nombre observé de neutrons émis ($\sim 10^6$) n'est pas d'origine thermonucléaire. Les auteurs exposent et analysent les données obtenues par l'étude des formes d'ondes électriques et des images photographiques prises à grande vitesse; ils montrent que des courants importants se forment à l'extérieur de la colonne centrale de plasma après la première contraction maximum. D'autres données semblent indiquer que ces courants passent dans la matière constitutive des parois chassée sous l'action du rayonnement provenant de la colonne centrale. Ces effets de paroi empêchent le chauffage par compression isentropique de se poursuivre et contaiminent également la colonne centrale de plasma; leur nature est telle qu'ils rendent impossible tout accroissement significatif de la température par une augmentation de la quantité d'énergie communiquée à la décharge.

Les auteurs concluent que c'est probablement à cause d'effets analogues que l'on n'a pas pu obtenir de températures thermonucléaires, au cours de nombreuses expériences de striction du plasma mettant en jeu de grandes énergies.

Distribution énergétique des neutrons produits au cours d'une reaction thermonucléaire, W. R. Faust et E. C. Harris *(U. S. Naval Research Laboratory, Washington, D. C., États-Unis d'Amérique)* Fusion Nucléaire **1**, 62 — 63 (1960)

Les auteurs de la monographie font le calcul de la distribution énergétique des neutrons au cours d'une reaction thermonucléaire. Ils partent de l'hypothèse que le mélange gazeux réagissant se trouve en état d'équilibre thermodynamique et que la section efficace différentielle relative à la production de neutrons est isotrope dans le système du centre de gravité.

АННОТАЦИИ ПО-РУССКИ

Осцилляции плазмы (I), И. Б. Бернштейн и С. К. Трехан *(Маттерхорнский проект, Принстонский университет, Принстон, Нью Джерси, США)*

Ядерный синтез **1**, 3-41 (1960)

Данная статья является первой частью обзора теории волновых явлений в плазмах. Разработана основная кинетическая теория с учетом кулоновских столкновений. Из этой теории выводятся магнитогидродинамические уравнения. Показано, что можно обращаться с замкнутой подсистемой этих уравнений или в пределах явлений доминирующих столкновений или в случаях, когда эффективные фазовые скорости интересующих явлений значительно больше тепловых скоростей. При рассмотрении доминирующих плазменных столкновений обсуждается теория транспортных коэффициентов.

Затем эти уравнения применяются к интенсивному исследованию волновых явлений с малой амплитудой в плазмах. Приводится рассмотрение вопроса о влиянии рассеяния на гидромагнитные волны. Гидромагнитные волны рассматриваются также с точки зрения теории Чью, Гольдберга и Лоу. Обсуждаются также продольные и поперечные осцилляции в токонесущих плазмах.

Рассматриваются осцилляции цилиндрической плазмы и обсуждается явление ионно-циклотронного резонанса. Показана возможность получения излучения в результате осцилляции плазмы в однородной сфере. Приводятся некоторые общие результаты стабильности продольных электронных осцилляций в неоднородных плазмах.

Кратко обсуждается трактовка электронных осцилляций высокой амплитуды а также перерыв этих осцилляций как механизм рассеяния для организованного движения плазмы.

Часть II этой статьи, намеченная к опубликованию в последующем номере данного журнала, будет посвящена рассмотрению осцилляций плазмы непосредственно с точки зрения кинетической теории.

Неустойчивости Рейли-Тейлора в плазме, ускоряемой в магнитном поле, Т. С. Грин и Дж. Б. Ф. Ниблетт *Исследовательский центр по атомному оружию, Олдермастон, Беркшир, Англия)*

Ядерный синтез **1**, 42-46 (1960)

При изучении сжатия дейтериевой плазмы с помощью аксиального магнитного поля, образованного широкой катушкой с одним витком, была использована высокоскоростная фотография. В данном эксперименте индуктивность катушки настолько незначительна по сравнению с индуктивностью внешней батареи конденсаторов, что плазма течет с постоянным ускорением в 5×10^{12} см/сек².

Завихрения, образуемые на внешней поверхности плазмы, рассматриваются как магнитогидродинамическая неустойчивость Рейли-Тейлора, вызванная внутренним ускорением поверхности, разделяющей магнитное поле и плазму. Наблюдаемое возрастание скоростей соответствует простой теории. Показывается, что отношение диффузии магнитного поля и импульса определяет, является ли сопротивление плазмы или ее вязкость доминирующим механизмом для гашения коротковолновой неустойчивости. При температурах и плотностях этого эксперимента сопротивление является основной причиной гашения и длина волны «максимальной неустойчивости», подсчитанная на этой основе, соответствует наблюдаемой величине.

Необходимый критерий гидромагнитной стабильности для плазмы с осевой симметрией, Клод Мерсье *(Исследовательская группа Ассоциации Евратом — Комиссариат по атомной энергии — Фонтене-о-Роз, Франция).*

Ядерный синтез **1**, 47-53 (1960)

В этой работе устанавливается необходимое условие стабильности для плазмы с осевой симметрией. Оно соответствует перемещениям, локализованным по соседству с поверхностями постоянного давления, на которых замыкаются силовые магнитные линии. Этот критерий сводится к критерию Сайдема для случая цилиндрической симметрии.

Быстрое сжатие нагреваемой плазмы в линейном z пинче, Х. А. Б. Бодин, А. А. Ньютон и Н. Ж. Пикок *(Исследовательский центр по атомному оружию, Олдермастон, Беркшир, Англия)*

Ядерный синтез **1**, 54-61 (1960)

В докладе дается краткое описание электрических характеристик разрядов с высокой и низкой первоначальной интенсивностью роста тока и электрических измерений, используемых для определения выполненной работы по быстрому сжатию плазмы. Рассматривается вопрос об энергии частиц и магнитного поля в разряде для случаев с высокой и низкой первоначальной интенсивностью роста тока ($dI/dt\,(0)$) и для разрядов при наличии и в отсутствии аксиального магнитного поля (H_z). Показывается, что гораздо более эффективное нагревание газа происходит как при наличии, так и в отсутствие (H_z) при большой величине $(dI/dt\ (0)$. тя $dI/dt\ (0) \sim 1{,}5 \times 10^{12}$ ампер/сек. $(H_z = 0)$ 15% первоначальной энергии конденсаторов идет на нагревание газа и 10% на магнитное поле при первом максимальном сокращении; для $dI/dt \sim 10^{11}$ ампер/сек $(H_z = 0.)$ за тот же период времени 11% идет на поле и лишь 6,5% на газ. Передача энергии газа оказывается ограниченной задолго до достижения максимального значения тока, и первоначальное резкое сжатие не сопровождалось последующим нагревом изоэнтропического сжатия, которого можно было бы ожидать, так что температура не превышала порядка 300 эв. Отсюда следует, что большинство наблюдаемого выхода нейтронов ($\sim 10^9$) не термоядерного происхождения. Приводятся и анализируются данные, полученные по формам электрических волн и по высокоскоростным фотографиям; показывается, что после первого максимального сокращения возникают сильные токи вне центрального шнура плазмы. Дальнейшие данные подтверждают, что эти токи протекают в материале, выпаренном из стенок излучением центрального шнура плазмы. Это влияние стенок объясняет отсутствие дальнейшего изоэнтропического нагрева сжатием и приводит также к загрязнению центрального плазменного шнура; оно таково, что предотвращает любое заметное увеличение нагрева при росте поглощаемой энергии в разряде. В заключение говорится, что подобные эффекты, вероятно, являются причиной неудач достижения температуры термоядерных реакций во многих экспериментах по пинчу высокой мощности.

Распределение энергии нейтронов, образованных в результате термоядерной реакции, У. Р. Фауст и Е. Дж. Харрис *(Морская исследовательская лаборатория США, Вашингтон, Округ Колумбия, США)*

Ядерный синтез **1**, 62-63 (1960)

Подсчитывается распределение энергии нейтронов при термоядерной реакции. Предполагается, что реагирующая смесь газов находится в термодинамическом равновесии и что дифференциальное поперечное сечение для генерации нейтронов является изотропным в системе центра масс.

RESUMENES EN ESPAÑOL

Oscilaciones del plasma (I), I. B. Bernstein y S. K. Trehan *(Proyecto Matterhorn, Universidad de Princeton, Princeton, New Jersey, los Estados Unidos de Norte America)* Fusión Nuclear **1**,3 — 41 (1960)

Este artículo constituye la primera parte de un estudio sistemático de la teoría de los fenómenos de tipo ondulatorio en los plasmas. Los autores desarrollan la teoría cinética en que se basan, teniendo en cuenta las colisiones coulómbicas, y derivan de ella ecuaciones magnetohidrodinámicas. Demuestran que cabe emplear un sistema cerrado de ecuaciones, tanto hasta el límite en que predominan las colisiones como hasta el límite en que las velocidades de fase efectivas de los fenómenos de interés son muy superiores a las velocidades térmicas. Comentan la teoría de los coeficientes de transporte para el caso de plasmasen los que predominan las colisiones. A continuación aplican estas ecuaciones a un tratamiento matemático muy extenso de los fenómenos ondulatorios de pequeña amplitud que tienen lugar en un plasma. Analizan el efecto de disipación en las ondas hidromagnéticas, y estudian éstas también desde el punto de vista de la teoría de Chew, Goldberger y Low. Discuten las oscilaciones transversales y longitudinales en plasmas portadores de corriente.

Seguidamente estudian las oscilaciones de un plasma cilíndrico y discuten el fenómeno de resonancia ciclotronica de los iones. Muestran la posibilidad de que una esfera uniforme del plasma emita radiaciones por oscilación. Dan algunos resultados de carácter general sobre la estabilidad de las oscilaciones longitudinales de electrones en plasma no uniformes.

También comentan un tratamiento abreviado de las oscilaciones de gran amplitud de los electrones y la interrupcion de estas oscilaciones como mecanismo de disipación del movimiento de un plasma organizado.

La segunda parte del trabajo aparecerá en otro número de esta revista y tratará del estudio de las oscilaciones del plasma aplicando directamente la teoría cinética.

Inestabilidades de Rayleigh-Taylor en un plasma acelerado magnéticamente, T. S. Green y G. B. F. Niblett *(Atomic Weapons Research Establishment, Aldermaston, Berkshire, Reino Unido)* Fusión Nuclear **1**, 42 — 46 (1960)

Los autores han estudiado la compresión de un plasma de deuterio por un campo magnético axial producido por un solenoide de una sola espira de diámetro grande. En este experimento la inductancia del solenoide es pequeña comparada con la de batería de condensadores exteriores, de manera que la implosión del plasma se produce con una aceleración constante de 5×10^{12} cm/s^2. En la superficie exterior del plasma se producen irregularidades que se consideran como inestabilidades magnetohidrodinámicas de Rayleigh-Taylor, producidas por la aceleración centrípeta de la superficie de separación entre el campo magnético y el plasma. Se ha observado que la velocidad del proceso responde a suposiciones teóricas sencillas. En el artículo se demuestra que la relación entre la difucibilidad del campo magnético y la cantidad de movimiento determina si la resistividad del plasma o su viscosidad es el mecanismo dominante para amortiguar las inestabilidades de corta longitud de onda. En las condiciones de temperatura y densidad en que se efectuó el experimento, la resistividad constituye el principal factor de amortiguamiento y la longitud de onda de la "inestabilidad máxima" calculada sobre esta base coincide con el valor observado.

Un criterio necesario de estabilidad hidromagnética para un plasma de simetría axial, Claude Mercier *(Groupe de Recherches de l'Association Euraton — C. E. A., Fontenay-aux-Roses, Seine, Francia)* Fusión Nuclear **1**, 47 — 53 (1960)

En la memoria, el autor establece una condición necesaria para la estabilidad de un plasma de simetría axial. Esta condición corresponde a desplazamientos que quedan localizados en las proximidades de aquellas superficies a presión constante en las que las líneas de fuerza son cerradas. En el caso de una simetría cilíndrica, este criterio queda reducido al de Suydam.

Calentamiento de un plasma por compresión rápida en la constricción lineal con campo magnético axial, H. A. B. Bodin, A. A. Newton y N. J. Peacock *(Atomic Weapons Research Establishment, Aldermaston, Berkshire, Reino Unido)* Fusión Nuclear **1**, 54 — 61 (1960)

Los autores describen sucintamente las características eléctricas de descargas efectuadas con índices altos y bajos de crecimiento inicial de la corriente, y examinan las mediciones que se utilizan para determinar los efectos que tiene sobre el plasma una compresión rápida. Se estudia la energía de las partículas y la energía magnética resultantes de la descarga con respecto a índices altos y bajos de crecimiento inicial de la corriente ($dI/dt(0)$) y con respecto a descargas con campo magnético axial (H_z) y sin él. Para valores altos de $dI/dt(0)$ el calentamiento del gas es mucho mayor, tanto con un H_z como sin él. Para un $dI/dt(0)$ de aproximadamente $1{,}5 \times 10^{12}$ amp/s (campo $H_z = 0$), el 15 por ciento de la energía inicial de la batería de condensadores pasa a calentar el gas y un 10 por ciento al campo magnético al producirse la primera contracción máxima; para un $dI/dt \sim 10^{11}$ amp/s ($H_z = 0$), el 11 por ciento pasa al campo magnético y sólo un 6,5 por ciento al gas. La energía absorbida por el gas queda limitada bastante antes de que la corriente alcance su máximo, y la implosión inicial no va seguida de un calentamiento por compresión isentrópica de la magnitud que podía esperarse, de manera que la temperatura no rebasa nunca los 300 eV aproximadamente; de ello se deduce que la mayor parte de la cantidad de neutrones emitidos ($\sim 10^6$) no es de origen termonuclear. Los autores exponen y analizan datos obtenidos de las formas de onda y de fotografías ultrarápidas, indicando cómo después de la primera contracción máxima se forman grandes corrientes fuera de la columna plasmática central. Otras pruebas sugieren que esas corrientes pasan a material de las paredes desprendido por la acción de las radiaciones de la columna central. A este efecto de las paredes se debe el que no se produzca un ulterior calentamiento por compresión isentrópica, así como la contaminación de la columna plasmática central; su naturaleza es tal que impide todo aumento importante del calentamiento cuando se aplica mayor energía a la descarga. Los autores llegan a la conclusión de que es a efectos análogos a los que probablemente se debe la imposibilidad de alcanzar una temperatura propia de las reacciones de fusión en muchos experimentos realizados sobre la constricción aplicando elevada energía.

Distribución energética de los neutrones producidos en une reacción termonuclear, W. R. Faust y E. G. Harris *(U. S. Naval Research Laboratory, Washington, D. C., Los Estados Unidos de Norte America)* Fusión Nuclear **1**, 62 — 63 (1960)

Los autores calculan la distribución energética de los neutrones producidos en una reacción termonuclear. Al efectuar el cálculo, se supone que la mezcla reactiva de gases se encuentra en equilibrio termodinámico y que la sección eficaz diferencial correspondiente a la producción de neutrones es isotrópica en el sistema del centro de gravedad.

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b) Todas las *ecuaciones* se numerarán con *números arábigos* colocados entre *paréntesis* a la derecha de cada ecuación. En la medida de lo posible, las ecuaciones se reproducirán exactamente en todas las traducciones; por ello, conviene emplear los símbolos reconocidos internacionalmente y evitar el uso de abreviaturas que sólo tengan sentido en un determinado idioma (por ejemplo, V_{med}, t_{max}, etc.). Deberán utilizarse símbolos simples cuyo significado se definirá en el texto.

c) *Figuras:*

   1) Las figuras se presentarán en *hojas aparte* y se numerarán con *números arábigos;* por ejemplo: «Fig. 1»;

   2) Los textos que acompañen a las figuras se agruparán en una *hoja aparte* con la numeración correspandiente;

   3) Sólo se enviarán fotografías cuando sea indispensable;

   4) Se recomiende que las figuras se hagan en dibujo lineal, utilizando en lo posible tinta negra y papel de dibujo blanco;

   5) Para facilitar el trabajo de traducción conviene que en las figuras sólo se empleen símbolos reconocidos internacionalmente; estos símbolos se definirán en el texto que vaya al pie de la figura.

d) Las *notas de pie de página* se indicarán en el texto con *números arábigos* y se agruparán en una hoja aparte.

e) Las *fuentes bibliográficas* se indicarán en el texto con *números arábigos* colocados entre *corchetes,* y se agruparán en una *hoja aparte* en orden numérico. Las menciones se harán de conformidad con los siguientes ejemplos: «[1] GRANADOS, A. J., *Rev. esp. fis.* 17, 483 1958)» o «[2] JONES, L. M., Plasma Physics. (XYZ Book Co., New York, 1957) 59».

f) Los *cuadros* se numerán con *números romanos* y los textos explicativos se agruparán en una *hoja aparte.* Siempre que sea posible se utilizarán en las columnas símbolos aceptados internacionalmente, dando su significado en los textos explicativos.

g) Junto con el original se enviará un *resumen* (de unas 300 palabras). El resumen se publicará en la revista en los cuatro idiomas de trabajo del Organismo.

Nota:

Deberán enviarse un original y dos copias de las figuras y de los cuadros, así como de las hojas separadas que contengan las notas de pie de página y las fuentes bibliográficas.

Al enviar el original se indicará el nombre y la dirección exacta de la persona a la que deberán remitirse las pruebas de imprenta. Con las pruebas se remitirá un formulario para encargar las separatas.

Los autores de los artículos publicados en la revista recibirán gratuitamente 50 separatas. Si desean recibir una cantidad mayor *habrán de* indicarlo al devolver las pruebas de imprenta.

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